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UNIFIED HYBRID CENSORING FOR THE XGAMMA DISTRIBUTION: FREQUENTIST, BOOTSTRAP, AND BAYESIAN ESTIMATION WITH INDUSTRIAL APPLICATIONS

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ABSTRACT

In spite of the growing interest in the Xgamma distribution as a versatile one-parameter lifetime distribution, the inferential properties of the Xgamma distribution under unified hybrid censoring (UHC) have not been studied. In this paper, we develop both classical and Bayesian inference procedures for the Xgamma distribution under unified hybrid censoring. We compute the maximum likelihood estimates using the BFGS optimization algorithm after a logarithmic transformation of the likelihood function. We also develop asymptotic normal, asymptotic normal-log, percentile bootstrap, and bootstrap-t confidence interval methods for interval estimation. In addition, we develop a random walk Metropolis–Hasting's algorithm using a logarithmic transformation and Robbins–Monro adaptation for the Bayesian inference under two gamma priors: a noninformative "Gamma"(0.001,0.001) and a data-adaptive "Gamma"(1,1/θ ̂_"MLE"). In the simulation study, we have conducted nine plans under three groups of design with 200 replicates each. We have demonstrated that the RMSE of the estimator θ ̂reduces from 0.1181 to 0.1048 when the sample size increases from n=20 to n=50, which is a reduction of 11.3%. In addition, the Bayesian estimator under Prior 2 has the least RMSE in all the plans. We have also demonstrated that the coverage probabilities of the ACI-NL interval lie in the range (0.510, 0.750) and perform better than other methods of interval estimation. We have also provided two examples using the failure data of Kevlar 49/epoxy strand and petroleum refinery equipment failure data. In these examples, the MLE and the Bayesian estimates of the parameters under the gamma prior lie within a relative error of 0.35%. We have also conducted the Kolmogorov–Smirnov test to check the goodness of fit of the Xgamma distribution to the data and have obtained p=0.0035 and p=0.0113 for the first and second datasets, respectively. We have used the criteria of AIC, BIC, and DIC to select the best model.

KEYWORDS: Xgamma Distribution, Unified Hybrid Censoring, Maximum Likelihood Estimation, Bayesian Inference, MCMC, Bootstrap Confidence Intervals, Reliability Function, Hazard Rate.

MSC Codes: 62N01, 62N02, 62F10, 62F15, 62N05, 62P30

1. INTRODUCTION

Lifetime data is commonly used in reliability engineering, biomedical research, and industrial quality control, where investigators measure the time to occurrence of a particular event for a given unit or subject [1, 2]. For many practical situations, it is often impossible to observe failure because of cost factors. Censoring schemes provide a reasonable approach to terminate an experiment early while retaining enough data to make statistical inference [3]. Of all censoring schemes, hybrid censoring is a combination of Type-I and Type-II censoring schemes that allows investigators to design more flexible experiments [2].

The unified hybrid censoring scheme (UHC), which includes other censoring schemes as special cases, was introduced to overcome some of the limitations of conventional hybrid censoring by introducing two-time limits, T_1 and T_2, where $T_1 < T_2$, along with a specified number of failures r [4, 5]. For UHC, the censoring time T_{eff} is adaptive depending on whether the r-th failure happens before T_1, within (T_1, T_2), or after T_2. The minimum time is guaranteed, while the experiment is restricted to a certain time limit. There are many studies on UHC for particular lifetime distributions. For instance, Jeon and Kang [4] used UHC for the Rayleigh distribution, while Nagy et al. [5] used UHC for Type-II unified hybrid censoring. Sarkar and Tripathy [6] used UHC for the inverse Gaussian distribution, while Ye et al. [7] used UHC for a class of inverted exponentiated distributions with competing causes of failure. More recent studies by Li et al. [8] and by Ahmad et al. [9] used UHC for progressive stress-accelerated life testing and parametric lifetime distribution, respectively.

Simultaneously, the Xgamma distribution, as proposed by Sen et al. [10], has attracted much attention as a single-parameter lifetime distribution. The Xgamma distribution is proposed as a finite mixture of exponential (θ) and gamma $(3, \theta)$ distributions. The Xgamma distribution is more flexible than the exponential distribution while retaining analytical ease through its single parameter $\theta > 0$. The Xgamma distribution has been extended to its inverse distribution [11], truncated Xgamma distribution [12], record-based Xgamma distribution [13], adaptive progressive censoring with competing risks [14], and an exhaustive review of its derivative forms [15].

Despite all these parallel developments, no work has explored the Xgamma distribution within the framework of a unified hybrid censoring scheme. The reason for this is critical because the unified hybrid censoring scheme offers a flexible censoring scheme that is highly desirable for reliability testing, where time as well as cost constraints are present. The gap requires developing frequentist as well as Bayesian approaches to estimate the parameters of the Xgamma distribution within the framework of UHC, investigating its finite-sample performance through simulations, and validating it on real reliability testing data.

The contributions of the current work can be summarized as follows: First, the likelihood function of the Xgamma distribution under unified hybrid censoring (UHC) is derived, and the maximum likelihood estimates of the parameters are obtained using the BFGS optimization algorithm along with a log-transformation to ensure positive estimates of the parameters. Second, the construction of different confidence intervals, namely asymptotic normal intervals (ACI-NA), asymptotic normal-log intervals (ACI-NL), percentile bootstrap confidence intervals, and bootstrap-t confidence intervals, is provided. Third, the Bayesian inference of the parameters of the Xgamma distribution under unified hybrid censoring is developed using the random walk Metropolis–Hasting's algorithm along with the Robbins–Monro adaptation on the log-scale [16, 17]. Fourth, a simulation study consisting of nine plans, divided into three groups, is conducted to examine the proposed estimators. Lastly, the proposed methodology is applied to two reliability engineering examples.

The rest of the paper is organized as follows: Section 2 introduces the Xgamma distribution and the unified hybrid censoring scheme. Section 3 presents the maximum likelihood estimation and the construction of the confidence intervals. Section 4 presents the Bayesian estimation methodology. Section 5 presents the simulation study. Section 6 presents the applications of the proposed methodology to two reliability engineering examples. Section 7 concludes the paper by providing a brief summary of the contributions and some directions for future research.

2. THE MODEL

The Xgamma distribution, as proposed in Sen et al. [10], represents a one-parameter continuous distribution on the positive real axis. The distribution is obtained as a mixture of an exponential distribution with rate parameter θ and a gamma distribution with shape parameter 3 and rate parameter θ, with weights $\theta/(1+\theta)$ for the exponential distribution and $1/(1+\theta)$ for the gamma distribution. The probability density function (PDF) of a random variable X following the Xgamma distribution with parameter $\theta > 0$ is given by:

$$f(x;\theta) = \frac{\theta^2}{1+\theta}\left(1+\frac{x^2}{2}\right)e^{-\theta x}, \quad x > 0.$$

The corresponding cumulative distribution function (CDF) is

$$F(x;\theta) = 1 - \left(1+\frac{\theta x(2+\theta x)}{2(1+\theta)}\right)e^{-\theta x}, \quad x > 0.$$

The survival (reliability) function is

$$S(x;\theta) = 1 - F(x;\theta) = \left(1+\frac{\theta x(2+\theta x)}{2(1+\theta)}\right)e^{-\theta x}$$

and the hazard (failure rate) function is

$$h(x;\theta) = \frac{f(x;\theta)}{S(x;\theta)} = \frac{\theta^2(1+x^2/2)}{1+\theta+\theta x(2+\theta x)/2}.$$

The mean of the Xgamma distribution can be expressed as follows: $E[X] = (\theta^2+2\theta+2)/[\theta(1+\theta)]$, which approaches $1/\theta$ as θ approaches infinity, thus replicating the mean of the exponential distribution. The Xgamma distribution offers an increasing hazard rate for all values of $\theta > 0$, making it a suitable model for components that demonstrate aging characteristics. The probability density function (PDF), cumulative distribution function (CDF), survival function, and hazard function for different values of θ are illustrated in Figure 1.

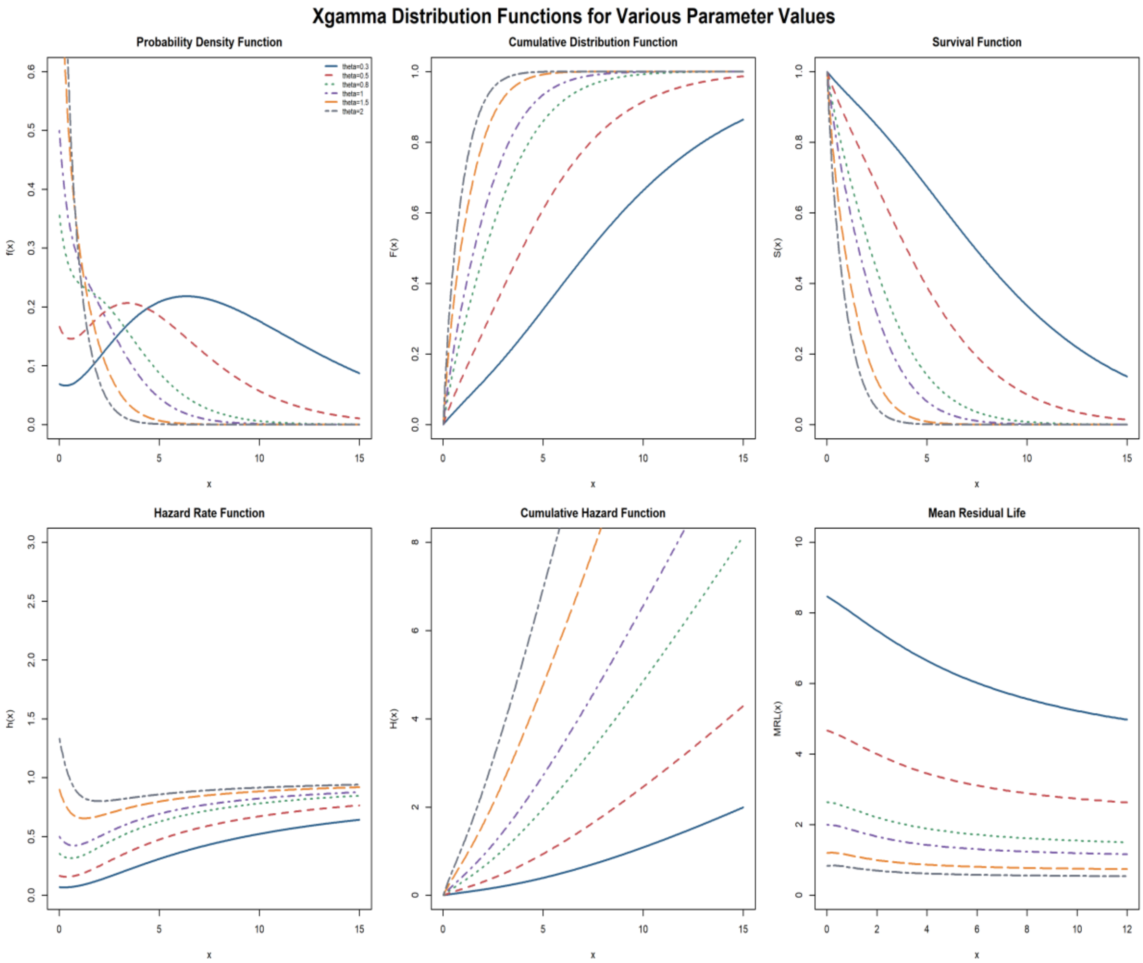

Figure 1: Xgamma Distribution Functions: PDF, CDF, Survival Function, And Hazard Function for Selected Values Of.

Under the unified hybrid censoring scheme, n identical units are subjected to a life test. The experimenter specifies three parameters for the experiment: the target failure number r $(1 \le r \le n)$, and two-time thresholds $T_1 < T_2$. Let $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ represent the order statistics of the lifetimes for the n units.

The effective censoring time T_{eff} is calculated as:

$$T_{\text{eff}} = \begin{cases} T_2, & \text{if } d < r \quad \text{(Case A: fewer than } r \text{ failures by } T_2\text{)}, \\ T_1, & \text{if } x_{r:n} \le T_1 \quad \text{(Case B: } r \text{ failures before } T_1\text{)}, \\ x_{r:n}, & \text{if } T_1 < x_{r:n} \le T_2 \quad \text{(Case C)}, \\ T_2, & \text{if } x_{r:n} > T_2 \quad \text{(Case D: } r\text{-th failure after } T_2\text{)}, \end{cases}$$

where $d = \#\{i: x_{i:n} \le T_2\}$ is the total number of failures observed by time T_2. The effective number of observed failures is $d_{\text{eff}} = \#\{i: x_{i:n} \le T_{\text{eff}}\}$, and the remaining $n - d_{\text{eff}}$ units are right-censored at T_{eff}. The likelihood function under UHC is

$$L(\theta|\mathbf{x}) = C\prod_{i=1}^{d_{\text{eff}}} f\,(x_{i:n};\theta)\cdot[S(T_{\text{eff}};\theta)]^{n-d_{\text{eff}}},$$

where C is a combinatorial constant that does not depend on θ. The log-likelihood function, ignoring the constant, is

$$\ell(\theta|\mathbf{x}) = \sum_{i=1}^{d_{\text{eff}}}\log f(x_{i:n};\theta) + (n-d_{\text{eff}})\log S(T_{\text{eff}};\theta).$$

3 MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimator (MLE) of θ is obtained by maximizing the log-likelihood function $\ell(\theta|\mathbf{x})$ as given in (7). The log-transformation $\phi = \log(\theta)$ is used with the constraint $\theta > 0$ to convert the constrained optimization problem into an unconstrained optimization problem. By replacing the Xgamma PDF given in (1) and the survival function given in (3) into (7), the explicit form of the log-likelihood function is obtained.

$$\ell(\theta|\mathbf{x}) = d_{\text{eff}}[2\log\theta - \log(1+\theta)] + \sum_{i=1}^{d_{\text{eff}}}\log\left(1+\frac{x_{i:n}^2}{2}\right) - \theta\sum_{i=1}^{d_{\text{eff}}} x_{i:n} + (n-d_{\text{eff}})\log S(T_{\text{eff}};\theta).$$

The score equation $\partial\ell/\partial\theta = 0$ does not have a closed-form solution. Therefore, numerical optimization is used with the BFGS algorithm, which is implemented in the maxLik R package. In order to avoid local optima in the optimization process, multiple initial values are used. As a last resort, a grid search is also implemented.

The observed Fisher information is calculated using $I(\hat\theta) = -\partial^2\ell/\partial\theta^2|_{\theta=\hat\theta}$, which gives the asymptotic standard error $\text{SE}(\hat\theta) = \sqrt{I^{-1}(\hat\theta)}$. For the reliability function $R(x_0) = S(x_0;\theta)$ and the hazard function $h(x_0)$ for a specified mission time x_0, the delta method [18] is used to obtain the standard errors: $\text{SE}(g(\hat\theta)) = |g'(\hat\theta)|\,\text{SE}(\hat\theta)$, where g is the function of interest. The observed Fisher information matrix for censored data follows the standard principles [19, 1].

Four types of confidence intervals are developed for the estimation of θ and the derived parameters $R(x_0)$ and $h(x_0)$. The asymptotic normal confidence interval (ACI-NA) at confidence level $1-\alpha$ is given by $\hat\theta \pm z_{\alpha/2}\,\text{SE}(\hat\theta)$. Here, $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution. The asymptotic normal-log confidence interval (ACI-NL) uses the logarithmic transformation to ensure the positive value of the confidence interval; it is given by $\hat\theta\exp(\pm z_{\alpha/2}\,\text{SE}(\hat\theta)/\hat\theta)$. The parametric bootstrap confidence interval is obtained using the resampling method from the fitted Xgamma distribution using the UHC method. The percentile interval is given by the $\alpha/2$ and $1-\alpha/2$ quantiles of the $B = 500$ bootstrap maximum likelihood estimates (MLEs) [20]. The bootstrap-t confidence interval uses the variance-stabilizing transformation; the logarithmic transformation is used when $\theta > 0$, and the logit transformation is used when $R(x_0) \in (0,1)$, using the methods developed by Hall [21] and Ahmed [22]. The entire MLE method is described in Algorithm 1.

Algorithm 1. Maximum Likelihood Estimation under Unified Hybrid Censoring

Require: Data $\mathbf{x} = (x_{1:n},\ldots,x_{d_{\text{eff}}:n})$, UHC parameters (n, r, T_1, T_2), tolerance $\epsilon = 10^{-6}$

Ensure: MLE $\hat\theta$, $\text{SE}(\hat\theta)$, $\hat R(x_0)$, $\hat h(x_0)$, confidence intervals

1. Determine T_{eff}: if $d < r$ then $T_{\text{eff}} = T_2$; if $x_{r:n} \le T_1$ then $T_{\text{eff}} = T_1$; if $T_1 < x_{r:n} \le T_2$ then $T_{\text{eff}} = x_{r:n}$; if $x_{r:n} > T_2$ then $T_{\text{eff}} = T_2$
2. Count $d_{\text{eff}} = \#\{i : x_i \le T_{\text{eff}}\}$
3. Construct $\ell(\theta|\mathbf{x}) = \sum_{i=1}^{d_{\text{eff}}}\log f(x_i;\theta) + (n-d_{\text{eff}})\log S(T_{\text{eff}};\theta)$
4. Apply log-transformation $\phi = \log(\theta)$ to enforce $\theta > 0$
5. Maximize $\ell(e^{\phi})$ via BFGS (maxLik); set $\hat\theta = e^{\hat\phi}$
6. Compute $I(\hat\theta) = -\partial^2\ell/\partial\theta^2|_{\hat\theta}$ numerically; $\text{SE}(\hat\theta) = \sqrt{I^{-1}(\hat\theta)}$
7. Derive $\hat R(x_0) = S(x_0;\hat\theta)$, $\hat h(x_0) = h(x_0;\hat\theta)$; SE via delta method
8. ACI-NA: $\hat\theta \pm z_{\alpha/2}\,\text{SE}(\hat\theta)$; ACI-NL: $\hat\theta\exp(\pm z_{\alpha/2}\,\text{SE}/\hat\theta)$

Return: $\hat\theta$, SE, $\hat R(x_0)$, $\hat h(x_0)$, ACI-NA, ACI-NL

Figure 2 presents the profile log-likelihood function for both datasets, illustrating the unimodality that ensures reliable convergence of the optimization procedure.

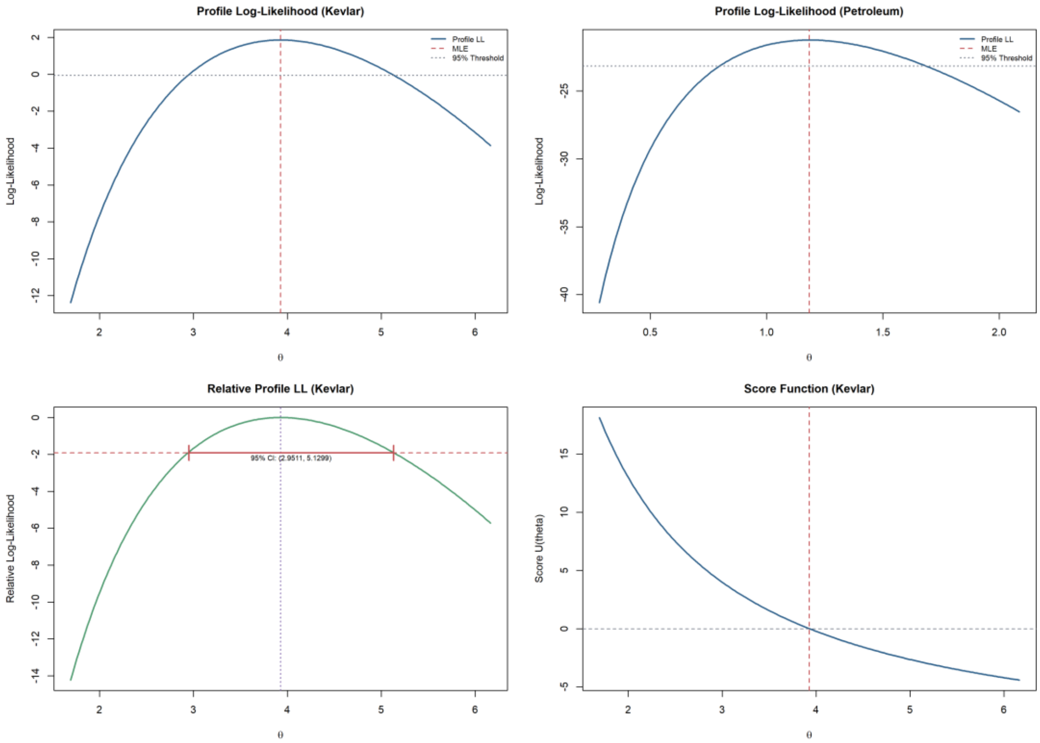

Figure 2: Profile Log-Likelihood Function for Under Unified Hybrid Censoring for Both Datasets.

4. BAYESIAN ESTIMATION

In the Bayesian approach, θ is treated as a random variable with a prior distribution $\pi(\theta)$ and a resulting posterior distribution $\pi(\theta|\mathbf{x}) \propto L(\theta|\mathbf{x})\pi(\theta)$. Since the posterior distribution of the Xgamma model under the UHC criterion does not have a closed form, MCMC methods are used to obtain samples from the posterior distribution $\pi(\theta|\mathbf{x})$.

In this study, two different gamma priors for the parameter θ are used. Prior 1: $\theta \sim \text{Gamma}(0.001, 0.001)$ — this is a non-informative gamma prior with a large variance, indicating that the data dominates the inference procedure. This gamma prior has been used by [23]. Prior 2: $\theta \sim \text{Gamma}(1, 1/\hat{\theta}_{\text{MLE}})$ — this is a data-adaptive gamma prior where the mean of the gamma distribution is equal to the MLE of the parameter θ. This gamma prior has a moderate variance. The use of two different gamma priors helps to examine the robustness of the results, which is a very important step in obtaining the final inference. This has been highlighted by [24, 25].

The type of Markov chain Monte Carlo (MCMC) sampler used is a random-walk Metropolis–Hasting's algorithm but is run on the log-scale to guarantee that the values of θ generated will be inherently positive. In the Metropolis–Hasting's algorithm, the current state is $\theta^{(t-1)}$. In the next iteration t, a new state is generated using the formula $\log\theta^{*} = \log\theta^{(t-1)} + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$. The acceptance ratio is calculated as $\alpha = \min(1, [\pi(\theta^{*}|\mathbf{x})\theta^{*}]/[\pi(\theta^{(t-1)}|\mathbf{x})\theta^{(t-1)}])$. In the algorithm, the variance σ^2 is updated during the burn-in period using the Robbins–Monro algorithm for stochastic approximation: $\log\sigma \leftarrow \log\sigma + \gamma_t(\alpha_t - 0.44)$, where $\gamma_t = t^{-0.6}$. This is done to target the theoretically optimal one-dimensional acceptance rate of 0.44 [16, 26, 27]. The theoretical justifications for adaptive MCMC algorithms are well established [17, 28].

The MCMC chain is run with $N = 50{,}000$ iterations, a burn-in period $B = 15{,}000$, and a thinning interval $k = 5$. The Geweke Z-test [29] is used to evaluate convergence, in which the means of the first 10% and last 50% of the chain, after the burn-in period, are compared. The effective sample size (ESS), using Geyer's initial positive sequence method [30], is also used to evaluate convergence. The point estimates are given by the posterior means and medians, while the interval estimates are given by the 95% Bayesian credible intervals (BCI), as well as the 95% highest posterior density (HPD) intervals. For the derived variables $R(x_0)$ and $h(x_0)$, samples are obtained using functional transformations of the samples of the parameter θ. The samples are given by $R^{(t)}(x_0) = S(x_0; \theta^{(t)})$ and $h^{(t)}(x_0) = h(x_0; \theta^{(t)})$. The Deviance Information Criterion (DIC) [31] is given by $\text{DIC} = -2\ell(\bar{\theta}) + 2p_D$, in which $p_D = 2[\ell(\bar{\theta}) - \overline{\ell(\theta)}]$.

The entire algorithm for the MCMC simulation is given in Algorithm 2.

Algorithm 2. Bayesian MCMC Estimation via Adaptive Random Walk Metropolis-Hastings

Require: Data $\mathbf{x}$, UHC parameters (r, T_1, T_2), prior $\pi(\theta)$, iterations $N = 50{,}000$, burn-in $B = 15{,}000$, thinning $k = 5$

Ensure: Posterior mean, median, SD, 95% BCI, 95% HPD, DIC

1. Initialize $\theta^{(0)} = \hat{\theta}_{\text{MLE}}$; proposal SD $\sigma = \text{SE}(\hat{\theta})$
2. **For** $t = 1$ to N **do**
3. Propose $\log\theta^* = \log\theta^{(t-1)} + \varepsilon$, $\varepsilon \sim N(0, \sigma^2)$
4. $\alpha = \min\left(1, \frac{\pi(\theta^*|\mathbf{x})\,\theta^*}{\pi(\theta^{(t-1)}|\mathbf{x})\,\theta^{(t-1)}}\right)$ where $\log\pi(\theta|\mathbf{x}) = \ell(\theta|\mathbf{x}) + \log\pi(\theta)$
5. Draw $u \sim U(0,1)$; if $u < \alpha$ set $\theta^{(t)} = \theta^*$, else $\theta^{(t)} = \theta^{(t-1)}$
6. **If** $t \leq B$ **then**
7. Adapt: $\log\sigma \leftarrow \log\sigma + t^{-0.6}(\alpha - 0.44)$ (Robbins-Monro)
8. **End If**
9. **End For**
10. Discard first B samples; retain every k-th $\Rightarrow$ 7,000 posterior draws
11. Convergence: Geweke Z-test (first 10% vs last 50%); ESS via Geyer IPS
12. Compute posterior mean, median, SD; 95% BCI from quantiles; 95% HPD
13. Derive $R^{(t)}(x_0) = S(x_0; \theta^{(t)})$, $h^{(t)}(x_0) = h(x_0; \theta^{(t)})$ for each retained sample
14. $\text{DIC} = -2\ell(\bar{\theta}) + 2p_D$, $p_D = 2[\ell(\bar{\theta}) - \overline{\ell(\theta)}]$

Return: Posterior summaries, credible intervals, DIC

5. SIMULATION STUDY

The performance of the proposed estimators in a finite sample is investigated through a simulation study. The true parameter is taken to be $\theta_0 = 0.5$. The data is simulated from the Xgamma distribution under the UHC using the inversion method described in Algorithm 3. The simulation study consists of nine plans grouped into three classes with varying sample size, censoring proportion, and time thresholds, as presented in Table 6.

Table 6: Simulation Study Plans with Varying Sample Sizes, Censoring Rates, And Time Thresholds Under Unified Hybrid Censoring.

Group	Plan	n	r	Cens. (%)	T_1	T_2	M
I: Varying n	I.1	20	15	25	3.97	8.30	200
	I.2	30	22	26.7	3.97	8.30	200
	I.3	50	35	30	3.97	8.30	200
II: Varying Cens.	II.1	30	27	10	3.97	8.30	200
	II.2	30	22	26.7	3.97	8.30	200
	II.3	30	18	40	3.97	8.30	200
III: Varying T_1, T_2	III.1	50	35	30	3.11	4.90	200
	III.2	50	35	30	3.97	6.63	200
	III.3	50	35	30	4.90	9.53	200

In Group I, the sample size is varied ($n = 20,30,50$). The censoring rate and time thresholds are approximately fixed. This allows us to study the impact of sample size on the accuracy of the estimator. In Group II, the sample size is fixed at $n = 30$. The censoring rate is varied at 10%, 26.7%, 40%. This allows us to study the impact of information loss on the accuracy of the estimator. In Group III, the sample size is fixed at $n = 50$, the censoring rate at 30%, and the time thresholds (T_1, T_2) vary in three settings: narrow, moderate, and wide. This allows us to study the impact of the location of the censoring window on the accuracy of the estimator. The experiments in each of these groups are run $M = 200$ times. The bootstrap resampling approach is encapsulated in Algorithm 3.

Algorithm 3. Parametric Bootstrap Confidence Intervals under Unified Hybrid Censoring

Require: MLEs $(\hat{\theta}, \hat{R}(x_0), \hat{h}(x_0))$, configuration (n, r, T_1, T_2), $B = 500$ replications

Ensure: Bootstrap SE, percentile CI, bootstrap-t CI for θ, $R(x_0)$, $h(x_0)$

1. Obtain $\hat{\theta}$ and $\text{SE}(\hat{\theta})$ from Algorithm 1
2. **For** $b = 1$ to B **do**
3. Generate $y_i = F^{-1}(u_i; \hat{\theta})$, $u_i \sim U(0,1)$, $i = 1, \ldots, n$ via numerical inversion
4. Apply UHC: sort $\{y_i\}$, determine T_{eff} from (r, T_1, T_2), extract failures $\mathbf{x}^{*(b)}$
5. Recompute $\hat{\theta}^{*(b)}$ via Algorithm 1; derive $\hat{R}^{*(b)}(x_0)$, $\hat{h}^{*(b)}(x_0)$
6. Compute $\text{SE}^{*(b)}$ via Fisher information on bootstrap sample
7. **End For**
8. $\text{SE}_{\text{boot}} = \sqrt{(B-1)^{-1}\sum_{b=1}^{B}(\hat{\theta}^{*(b)} - \bar{\theta}^*)^2}$
9. Percentile CI: $(\hat{\theta}^*_{(\alpha/2)}, \hat{\theta}^*_{(1-\alpha/2)})$
10. Boot-t: transform $\phi^{*(b)} = \log\hat{\theta}^{*(b)}$; $t^{*(b)} =$

$(\phi^{*(b)} - \hat{\phi})/\mathrm{SE}_{\phi}^{*(b)}$; invert quantiles

11. For $R(x_0) \in (0,1)$: use logit transform; back-transform via expit

Return: Bootstrap SE, percentile CI, bootstrap-t CI

There are two performance metrics used in point estimation, and they are the root mean squared error $\mathrm{RMSE} = \sqrt{M^{-1} \sum_{j=1}^{M} (\hat{\theta}_j - \theta_0)^2}$ and the mean absolute bias $\mathrm{MAB} = M^{-1} \sum_{j=1}^{M} |\hat{\theta}_j - \theta_0|$. In interval estimation, the average interval length (AIL) and the coverage probability (CP) at the nominal 95% level are reported. Table 7 provides a summary of the results obtained from the nine scenarios and the three estimation methods, including MLE, Bayes Prior 1, and Bayes Prior 2, for θ, $R(x_0)$, and $h(x_0)$.

Table 7: Root Mean Squared Error (RMSE) And Mean Absolute Bias (MAB) Of Parameter Estimates Across Simulation Plans.

Plan	Metric	θ MLE	θ Prior 1	θ Prior 2	$R(x)$ MLE	$R(x)$ Prior 1	$R(x)$ Prior 2	$h(x)$ MLE	$h(x)$ Prior 1	$h(x)$ Prior 2
I.1	RMSE	0.1181	0.1184	0.1158	0.1308	0.1310	0.1283	0.0853	0.0868	0.0848
	MAB	0.1050	0.1052	0.1027	0.1095	0.1097	0.1072	0.0735	0.0750	0.0731
I.2	RMSE	0.1146	0.1147	0.1131	0.1249	0.1249	0.1232	0.0823	0.0833	0.0820
	MAB	0.1024	0.1026	0.1011	0.1066	0.1066	0.1051	0.0717	0.0727	0.0716
I.3	RMSE	0.1048	0.1049	0.1039	0.1062	0.1062	0.1052	0.0725	0.0731	0.0723
	MAB	0.0964	0.0965	0.0956	0.0939	0.0939	0.0930	0.0653	0.0659	0.0653
II.1	RMSE	0.1223	0.1223	0.1206	0.1388	0.1389	0.1369	0.0891	0.0902	0.0888
	MAB	0.1136	0.1135	0.1119	0.1251	0.1252	0.1234	0.0814	0.0825	0.0812
II.2	RMSE	0.1128	0.1131	0.1113	0.1215	0.1217	0.1198	0.0803	0.0814	0.0800
	MAB	0.1015	0.1017	0.1001	0.1048	0.1050	0.1033	0.0707	0.0717	0.0705
II.3	RMSE	0.1016	0.1019	0.1003	0.0979	0.0979	0.0962	0.0690	0.0698	0.0686
	MAB	0.0878	0.0881	0.0865	0.0795	0.0794	0.0780	0.0577	0.0586	0.0575
III.1	RMSE	0.0988	0.0991	0.0979	0.0837	0.0836	0.0827	0.0630	0.0635	0.0627
	MAB	0.0885	0.0887	0.0877	0.0731	0.0730	0.0722	0.0557	0.0562	0.0555
III.2	RMSE	0.1077	0.1077	0.1069	0.1079	0.1078	0.1070	0.0742	0.0747	0.0741
	MAB	0.0984	0.0984	0.0976	0.0954	0.0953	0.0946	0.0666	0.0671	0.0666
III.3	RMSE	0.1066	0.1067	0.1056	0.1094	0.1094	0.1083	0.0741	0.0747	0.0740
	MAB	0.0983	0.0984	0.0974	0.0970	0.0970	0.0960	0.0670	0.0676	0.0669

From the results provided in Table 7, the following patterns can be established: In Group I, the RMSE of the estimator $\hat{\theta}$ is reduced when the sample size increases from $n = 20$ (Plan I.1) to $n = 50$ (Plan I.3) for the maximum likelihood estimator (MLE), from 0.1181 to 0.1048, i.e., by 11.3%. This pattern of reduction is monotonic and applies to all three methods of estimation and to the three quantities: θ, $R(x_0)$, and $h(x_0)$. In all plans, the Bayesian estimator under Prior 2 (data-adaptive) has the best RMSE, although the gains relative to the MLE and Prior 1 are small, i.e., 1–2%.

In Group II, the impact of the censoring level is not monotonic: the best RMSE of the estimator $\hat{\theta}$ occurs when the censoring level is the highest (40% in Plan II.3), due to the larger effective sample size of the observed failures when the experiment lasts longer before censoring. For the other two quantities, $R(x_0)$ and $h(x_0)$, the same pattern is observed: the best RMSE of 0.0979 and 0.0690 occurs when the censoring level is 40%, compared to 0.1388 and 0.0891 when the censoring level is 10%.

In Group III, the shortest time window (Plan III.1) has the best RMSE for the estimation of the quantities, while the moderate and wide windows have similar results.

Table 8 reports the average interval length and coverage probability for θ across the four interval methods.

Table 8: Average Interval Length (AIL) And Coverage Probability (CP) For $\boldsymbol{\theta}$ Across Simulation Plans and Confidence Interval Methods.

Plan	n	AIL ACI-NA	AIL ACI-NL	AIL BCI	AIL HPD	CP ACI-NA	CP ACI-NL	CP BCI	CP HPD
I.1	20	0.2785	0.2841	0.2778	0.2749	0.660	0.745	0.695	0.665
I.2	30	0.2285	0.2316	0.2279	0.2259	0.535	0.630	0.570	0.530
I.3	50	0.1806	0.1821	0.1803	0.1789	0.455	0.515	0.460	0.445
II.1	30	0.2098	0.2124	0.2095	0.2077	0.390	0.520	0.455	0.410
II.2	30	0.2290	0.2321	0.2286	0.2263	0.510	0.615	0.565	0.520
II.3	30	0.2544	0.2583	0.2531	0.2508	0.700	0.750	0.730	0.705
III.1	50	0.1957	0.1976	0.1955	0.1941	0.525	0.590	0.560	0.535

III.2	50	0.1807	0.1822	0.1802	0.1790	0.430	0.520	0.460	0.425
III.3	50	0.1793	0.1808	0.1791	0.1777	0.425	0.510	0.460	0.440

The results regarding the interval estimation methods, as given in Table 8, show that the coverage probabilities obtained using the ACI-NL method are the highest across all the plans, with the coverage probabilities varying from 0.510 (Plan III.2) to 0.750 (Plans II.3). Hence, the ACI-NL method may be recommended as the appropriate method for interval estimation of the parameter θ. Although the HPD method provides the minimum average interval length over all the plans, it provides the least coverage probabilities among the methods. The BCI method provides a compromise between the coverage probabilities and the interval lengths, and it provides the second highest coverage probabilities after the ACI-NL method. As may be noted from the results, the AIL coverage probabilities decrease with an increase in the sample sizes (Group I), and the results are as expected because the precision increases with the sample sizes.

The coverage probabilities also depend upon the level of censoring, and it may be noted that the coverage probabilities increase with the level of censoring (Group II).

Figure 3 displays boxplots of the parameter estimate across simulation replications, providing a visual summary of the bias and variability patterns.

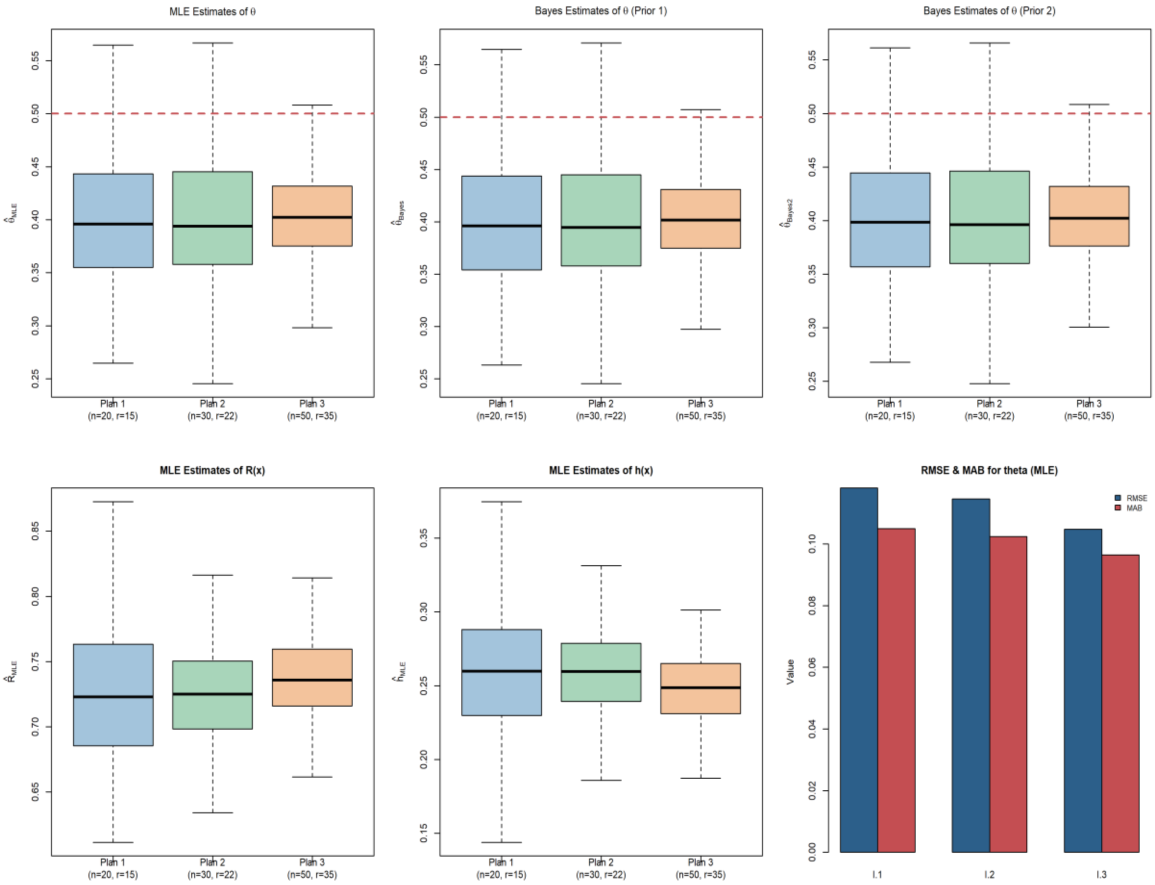

Figure 3: Boxplots Of Parameter Estimates (,,) Across Simulation Replications for Selected Plans.

Figure 4 provides an alternative visualization of the simulation results, comparing bias and MSE across the nine plans.

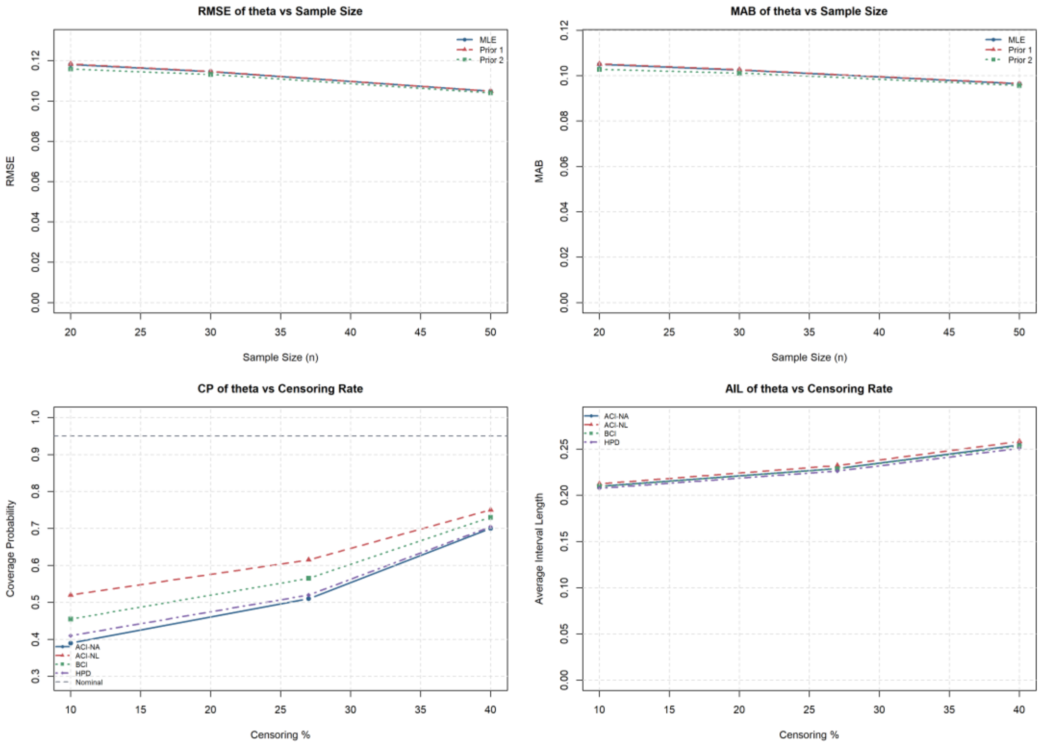

Figure 4: Bias And MSE Comparison Across Simulation Plans for MLE And Bayesian Estimators.

6. APPLICATIONS

The proposed methodologies are then applied to the following two real datasets used in reliability engineering. The first dataset consists of failure times in hours for $n = 49$ Kevlar 49/epoxy strand specimens tested at 90% stress level, originally reported in Andrews and Herzberg (1985). The second dataset consists of failure times in years for $n = 20$ petroleum refinery equipment units, originally reported in Lawless [1]. For each of these datasets, the UHC scheme is chosen with r being 70–75% of the sample size, T_1 at the 50th percentile, and T_2 at the 85th percentile of the failure times. This is because the experimenter in this scenario wishes to observe most of the failures within a moderate time period.

Table 2 summarizes the UHC configuration and descriptive statistics for both datasets. Both datasets fall into Case C of the UHC scheme, where the r-th failure occurs between T_1 and T_2.

Table 2: Data Summary: Unified Hybrid Censoring Configuration and Descriptive Statistics for Both Datasets.

Statistic	Kevlar 49/Epoxy	Petroleum Equipment
Panel A: Censoring Configuration		
Sample size (n)	49	20
Target failures (r)	35	15
T_1	0.2300	1.1200
T_2	0.6460	2.0030
Effective time (T_{eff})	0.4300	1.6200
Observed failures (d_{eff})	35	15
Censored observations	14	5
Censoring rate (%)	28.6	25
UHC Case	C	C
Panel B: Descriptive Statistics (Observed Failures)		
Minimum	0.0100	0.1900
Maximum	0.4300	1.6200
Mean	0.1603	0.8600
Median	0.1100	0.8100
Std. Dev.	0.1343	0.4756
Skewness	0.7550	0.1440
Kurtosis	-0.9587	-1.4925

Q_1	0.0550	0.4750
Q_3	0.2400	1.2300
IQR	0.1850	0.7550
Panel C: Mission Time		
Mission time (x_0)	0.1100	0.8100

Table 1 presents the ordered failure times along with the estimated CDF values under UHC for both datasets.

Table 1: Ordered Failure Times Under Unified Hybrid Censoring for Both Datasets.

i	x_i (Kevlar)	$\hat{F}(x_i)$ (Kevlar)		i	x_i (Petroleum)	$\hat{F}(x_i)$ (Petroleum)
1	0.0100	0.0307		1	0.1900	0.1099
2	0.0100	0.0307		2	0.2600	0.1454
3	0.0200	0.0602		3	0.3000	0.1647
4	0.0200	0.0602		4	0.3900	0.2058
5	0.0200	0.0602		5	0.5600	0.2764
6	0.0300	0.0887		6	0.6300	0.3033
7	0.0300	0.0887		7	0.7400	0.3434
8	0.0400	0.1160		8	0.8100	0.3677
9	0.0500	0.1424		9	0.9200	0.4043
10	0.0600	0.1677		10	1.0700	0.4514
11	0.0700	0.1922		11	1.1700	0.4811
12	0.0700	0.1922		12	1.2900	0.5151
13	0.0800	0.2158		13	1.4500	0.5579
14	0.0900	0.2385		14	1.5000	0.5707
15	0.0900	0.2385		15	1.6200	0.6004
16	0.1000	0.2605				
17	0.1000	0.2605				
18	0.1100	0.2816				
19	0.1100	0.2816				
20	0.1200	0.3021				
21	0.1300	0.3218				
22	0.1800	0.4112				
23	0.1900	0.4273				
24	0.2000	0.4430				
25	0.2300	0.4870				
26	0.2400	0.5008				
27	0.2400	0.5008				
28	0.2900	0.5638				
29	0.3400	0.6181				
30	0.3500	0.6281				
31	0.3600	0.6378				
32	0.3800	0.6564				
33	0.4000	0.6739				
34	0.4200	0.6906				
35	0.4300	0.6986				

The results section shows the maximum likelihood estimates, standard errors, and four types of confidence intervals. The summary of the results is provided in Table 3. For the Kevlar dataset, the estimate is $\hat{\theta} = 3.9282$ with SE = 0.5587. The reliability is then $\hat{R}(x_0) = 0.7184$ when $x_0 = 0.11$, which is the median failure time. The four confidence intervals for θ are similar to each other. The ACI-NL is (2.9725,5.1912), which is slightly larger than that of the ACI-NA (2.8331,5.0233). The asymmetry correction of the ACI is reflected by the logarithm transformation. For the Petroleum dataset, the estimate is $\hat{\theta} = 1.1840$ with SE = 0.2256. The reliability is then $\hat{R}(x_0) = 0.6323$ when $x_0 = 0.81$. The bootstrap-t intervals are wider than the other intervals for the Petroleum dataset. This is likely because of the larger tails of the bootstrap-t distribution.

Table 3: Maximum Likelihood Estimates with Standard Errors and Confidence Intervals Under Unified Hybrid Censoring.

Dataset	Parameter	MLE	SE	ACI-NA	ACI-NL	Boot-Perc	Boot-t

Kevlar	$\hat{\theta}$	3.9282	0.5587	(2.8331, 5.0233)	(2.9725, 5.1912)	(2.9782, 5.1642)	(2.9774, 5.3000)
	$\hat{R}(x_0)$	0.7184	0.0404	(0.6392, 0.7975)	(0.6330, 0.7905)	(0.6337, 0.7897)	(0.6283, 0.7937)
	$\hat{h}(x_0)$	2.8466	0.4722	(1.9211, 3.7721)	(2.0565, 3.9402)	(2.0469, 3.8921)	(2.0151, 4.0211)
Petroleum	$\hat{\theta}$	1.1840	0.2256	(0.7418, 1.6262)	(0.8150, 1.7201)	(0.7076, 1.5696)	(0.8927, 2.0398)
	$\hat{R}(x_0)$	0.6323	0.0784	(0.4785, 0.7860)	(0.4702, 0.7691)	(0.5074, 0.8072)	(0.4630, 0.7742)
	$\hat{h}(x_0)$	0.5167	0.1132	(0.2948, 0.7387)	(0.3363, 0.7939)	(0.2720, 0.7037)	(0.3302, 0.8086)

Table 12 presents the bootstrap confidence intervals alongside the MCMC computational specifications used in the Bayesian analysis.

Table 12: Bootstrap Confidence Intervals and MCMC Computational Specifications.
Panel A: Bootstrap Confidence Intervals.

Dataset	Parameter	Estimate	Boot SE	Percentile CI	Boot-t CI
Kevlar	θ	3.9282	0.5927	(2.9782, 5.1642)	(2.9774, 5.3000)
	$R(x_0)$	0.7184	0.0423	(0.6337, 0.7897)	(0.6283, 0.7937)
	$h(x_0)$	2.8466	0.5005	(2.0469, 3.8921)	(2.0151, 4.0211)
Petroleum	θ	1.1840	0.2353	(0.7076, 1.5696)	(0.8927, 2.0398)
	$R(x_0)$	0.6323	0.0817	(0.5074, 0.8072)	(0.4630, 0.7742)
	$h(x_0)$	0.5167	0.1178	(0.2720, 0.7037)	(0.3302, 0.8086)

Panel B: Mcmc Computational Settings.

Setting	Value
Total iterations (N)	50,000
Burn-in	15,000
Thinning interval	5
Effective posterior samples	7,000
Proposal distribution	Log-Normal (RW)
Adaptation	Robbins-Monro
Target acceptance rate	0.44 (1D)
ESS method	Geyer IPS
Convergence test	Geweke Z-test
Bootstrap replications (B)	500
Prior 1	$\theta \sim$ Gamma($0.001, 0.001$)
Prior 2	$\theta \sim$ Gamma($1, 1/\hat{\theta}_{\mathrm{MLE}}$)

From the convergence diagnostics of the Markov chain Monte Carlo (MCMC) procedures, as provided in Table 4, we note that convergence is achieved for all the chains, i.e., the two datasets and the two priors used. This is due to the fact that the acceptance rates lie within the range of 0.449 and 0.459, which is very close to the theoretical optimum of 0.44 for one-dimensional random walk Metropolis–Hasting's algorithms. Furthermore, the effective sample sizes are greater than 5,400 for all the chains, which is significantly greater than the recommended minimum requirement. Additionally, the Geweke Z-tests for convergence are non-significant ($p > 0.54$).

Table 4: MCMC Convergence Diagnostics for Bayesian Estimation Under Both Prior Specifications.

Dataset	Prior	Accept. Rate	ESS (θ)	Geweke Z	Geweke p	Converged
Kevlar	Prior 1	0.455	5488	-0.4491	0.6533	Yes
	Prior 2	0.449	5494	-0.6065	0.5442	Yes
Petroleum	Prior 1	0.459	5630	-0.1708	0.8644	Yes
	Prior 2	0.459	5864	-0.5675	0.5704	Yes

Table 5 displays the Bayesian posterior estimates using the two priors. The posterior means for θ in the Kevlar dataset are 3.9391 (Prior 1) and 3.9322 (Prior 2), with differences of only 0.28% and 0.10% compared to the maximum likelihood estimate (MLE), respectively. The good agreement between the Bayesian estimates and the classical maximum likelihood estimates suggests that the data are informative enough to override the prior, which is a good property in practice. The length of the 95% HPD intervals is consistently shorter than the length of the corresponding 95% BCI intervals for all parameters in all datasets. This is expected, as HPD intervals are optimal in terms of being the shortest-length credible intervals. In the Petroleum dataset, the posterior means for θ using Prior 1 (1.1881) and Prior 2 (1.1870) are similarly in good agreement with the maximum likelihood estimate (1.1840), with differences of less than 0.35%.

Table 5: Bayesian Posterior Estimates Under Two Prior Specifications for Both Datasets.

Dataset	Prior	Parameter	Mean	Median	SD	95% BCI	95% HPD
Kevlar	Prior 1	θ	3.9391	3.9127	0.5589	(2.9086, 5.0858)	(2.9298, 5.1063)
		$R(x_0)$	0.7185	0.7195	0.0401	(0.6388, 0.7951)	(0.6372, 0.7932)
		$h(x_0)$	2.8564	2.8335	0.4718	(1.9887, 3.8258)	(2.0064, 3.8431)
	Prior 2	θ	3.9322	3.8989	0.5539	(2.9251, 5.0835)	(2.9343, 5.0869)
		$R(x_0)$	0.7190	0.7205	0.0397	(0.6389, 0.7939)	(0.6387, 0.7931)
		$h(x_0)$	2.8505	2.8218	0.4676	(2.0025, 3.8239)	(2.0102, 3.8267)
Petroleum	Prior 1	θ	1.1881	1.1713	0.2270	(0.7933, 1.6709)	(0.7564, 1.6312)
		$R(x_0)$	0.6337	0.6367	0.0771	(0.4779, 0.7749)	(0.4758, 0.7723)
		$h(x_0)$	0.5168	0.5104	0.1125	(0.3163, 0.7505)	(0.3200, 0.7538)
	Prior 2	θ	1.1870	1.1721	0.2241	(0.7877, 1.6626)	(0.7807, 1.6459)
		$R(x_0)$	0.6340	0.6364	0.0763	(0.4802, 0.7770)	(0.4850, 0.7797)
		$h(x_0)$	0.5163	0.5107	0.1112	(0.3134, 0.7467)	(0.3098, 0.7391)

Figure 5 displays the MCMC trace plots, posterior density plots, and autocorrelation functions, confirming the visual impression of convergence and good mixing.

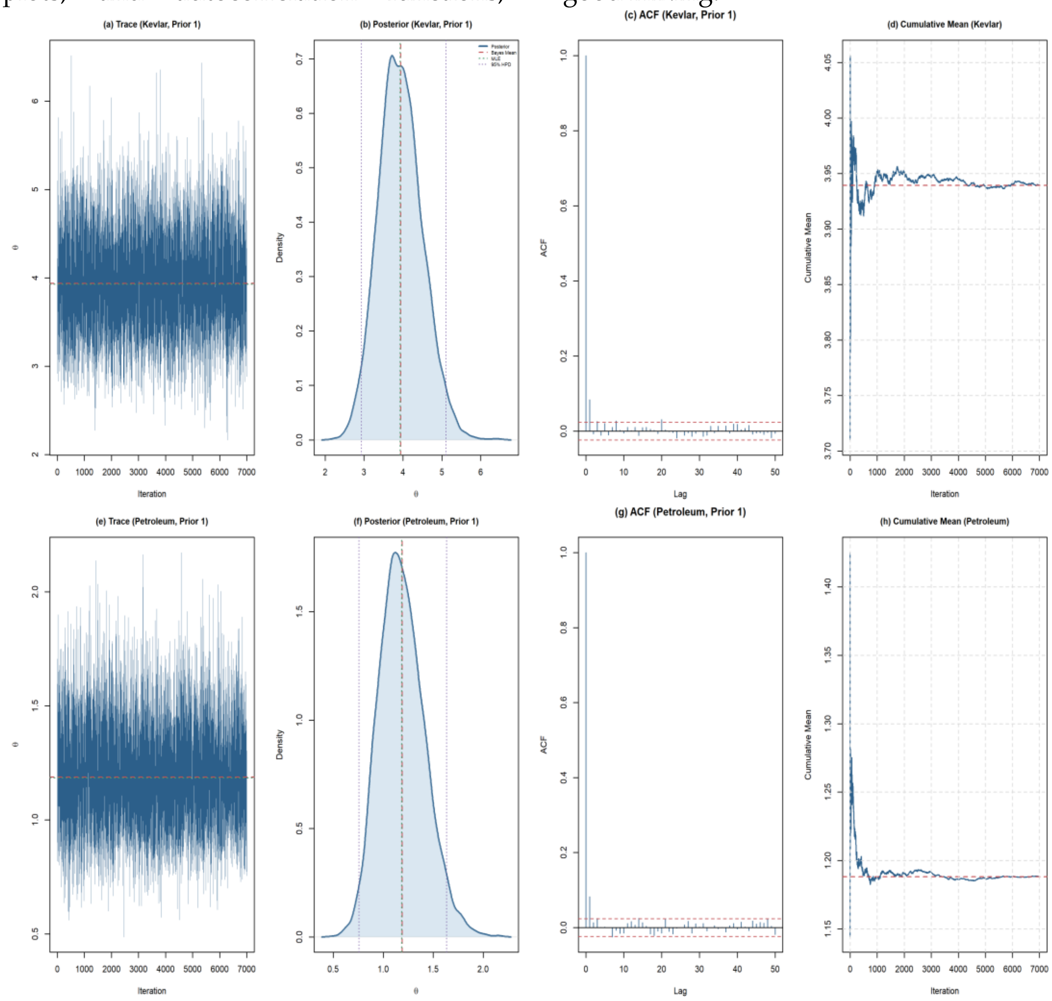

Figure 5: MCMC Diagnostics: Trace Plots, Posterior Densities, And Autocorrelation Functions for Under Both Priors and Both Datasets.

Table 10 shows the agreement between maximum likelihood estimates (MLE) and Bayesian estimates by reporting relative differences. For all six parameter-data set pairs, the maximum relative difference is 0.35% for $\hat{\theta}$ in the Petroleum data set under Prior 1, which shows that the prior has little effect on the posterior under moderately informative data. This is an advantage of the methodology, which can use either prior.

Table 10: Comparison Of MLE And Bayesian Estimates with Relative Differences (%).

Dataset	Parameter	MLE	Bayes P1	Bayes P2	RD (%) P1	RD (%) P2
Kevlar	θ	3.9282	3.9391	3.9322	0.28	0.10
	$R(x_0)$	0.7184	0.7185	0.7190	0.02	0.09
	$h(x_0)$	2.8466	2.8564	2.8505	0.35	0.14
Petroleum	θ	1.1840	1.1881	1.1870	0.35	0.25
	$R(x_0)$	0.6323	0.6337	0.6340	0.23	0.28
	$h(x_0)$	0.5167	0.5168	0.5163	0.02	0.08

Figure 6 compares the posterior distributions under the two prior specifications, illustrating their near-identical shapes.

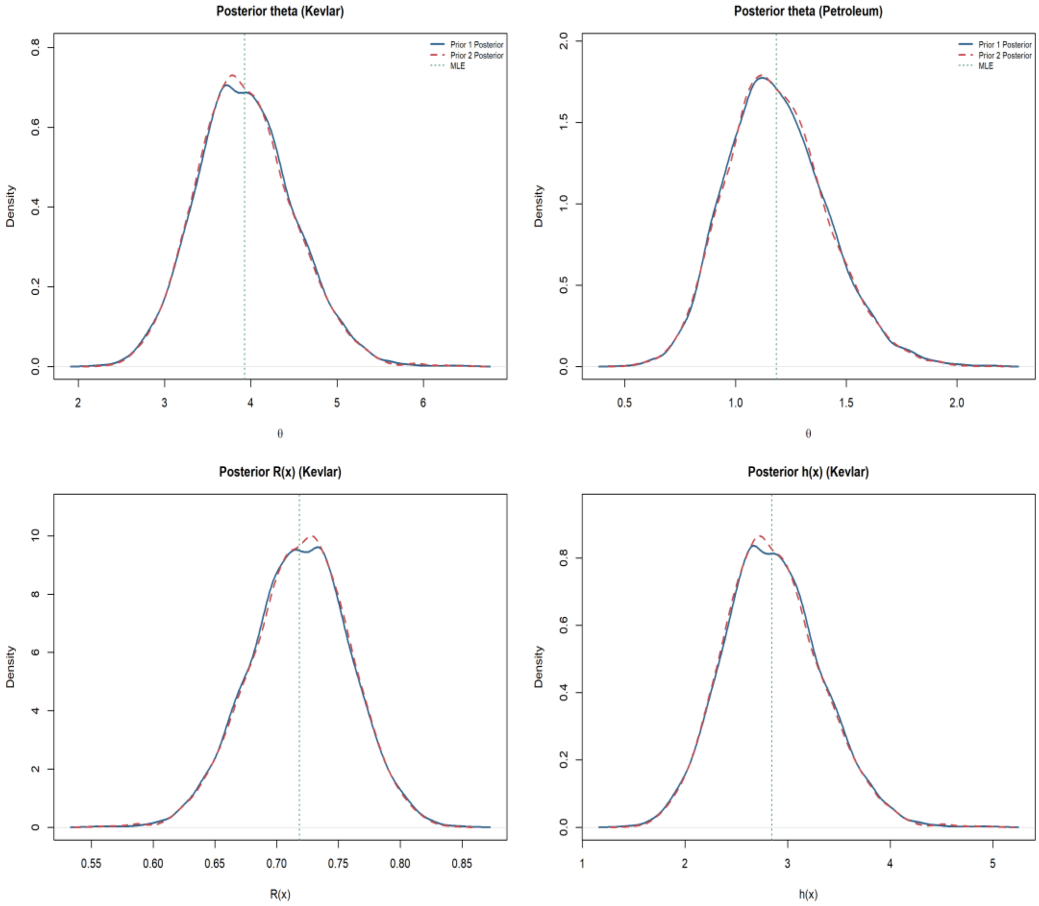

Figure 6: Comparison Of Posterior Distributions for Under Prior 1 And Prior 2 For Both Datasets.

Table 11 provides a comprehensive summary of the real data analysis, including parameter estimates from all four methods (MLE, Bayes Prior 1, Bayes Prior 2, Bootstrap) and model selection criteria.

Table 11: Comprehensive Real Data Analysis: Parameter Estimation and Model Selection Criteria.
Panel A: Parameter Estimation

Dataset	Method	$\hat{\theta}$	$\hat{R}(x_0)$	$\hat{h}(x_0)$	SE(θ)
Kevlar	MLE	3.9282	0.7184	2.8466	0.5587
	Bayes P1	3.9391	0.7185	2.8564	0.5589
	Bayes P2	3.9322	0.7190	2.8505	0.5539

	Bootstrap	3.9282	0.7184	2.8466	0.5927
Petroleum	MLE	1.1840	0.6323	0.5167	0.2256
	Bayes P1	1.1881	0.6337	0.5168	0.2270
	Bayes P2	1.1870	0.6340	0.5163	0.2241
	Bootstrap	1.1840	0.6323	0.5167	0.2353

Panel B: Model Selection Criteria

Dataset	Criterion	Value
Kevlar	-2ℓ	-3.72
	AIC	-1.72
	BIC	-0.16
	AICc	-1.59
	HQIC	-1.18
	CAIC	0.84
	DIC (P1)	-1.71
	DIC (P2)	-1.75
Petroleum	-2ℓ	42.49
	AIC	44.49
	BIC	45.20
	AICc	44.80
	HQIC	44.49
	CAIC	46.20
	DIC (P1)	44.51
	DIC (P2)	44.48

The model selection criterion values provided in Panel B of Table 11 show that the Kevlar dataset's AIC and BIC have negative values of -1.72 and -0.16, respectively. This is due to the small values of the observed failure times, with most of the values being below 0.5. The DIC values for the Kevlar dataset using both priors are -1.71 and -1.75, respectively, which are in agreement with the AIC. The values of the AIC (44.49), BIC (45.20), and DIC values (44.51, 44.48) for the Petroleum dataset are in good agreement with one another. The DIC value for Prior 2 is slightly lower than that of Prior 1. This implies that the data-adaptive prior specification seems to have a slightly better fit to the Petroleum dataset.

The goodness-of-fit test results for the Xgamma distribution applied to the two datasets are provided in Table 9. The test results show that the null hypothesis is rejected at $\alpha = 0.05$ for both datasets using the Kolmogorov–Smirnov test. The p-values are 0.0035 for the Kevlar dataset and 0.0113 for the Petroleum dataset. The Anderson–Darling test and the Cramer–von Mises test also show that the null hypothesis can be rejected at $\alpha = 0.05$ [32, 33]. The implication of these test results is that, while the Xgamma distribution provides a reasonable first approximation to the two datasets, the fit of the distribution to the two datasets is not statistically significant. This is not unexpected, given the one-parameter distributions' application to complex failure data.

Table 9: Goodness-Of-Fit Test Results for the Xgamma Distribution Fitted to Both Datasets.

Test Statistic	Kevlar 49/Epoxy	Petroleum Equipment
KS Statistic	0.3014	0.3996
KS p-value	0.0035	0.0113
AD Statistic	6.3223	2.6189
CvM Statistic	1.2467	0.5335
Decision ($\alpha = 0.05$)	Reject H_0	Reject H_0

Note: Dataset 1: Rejected ($p = 0.003$); Dataset 2: Rejected ($p = 0.011$). Results Should Be Interpreted with Caution.

Figure 7 displays the estimated survival curves overlaid with the empirical survival function, providing a visual assessment of model fit.

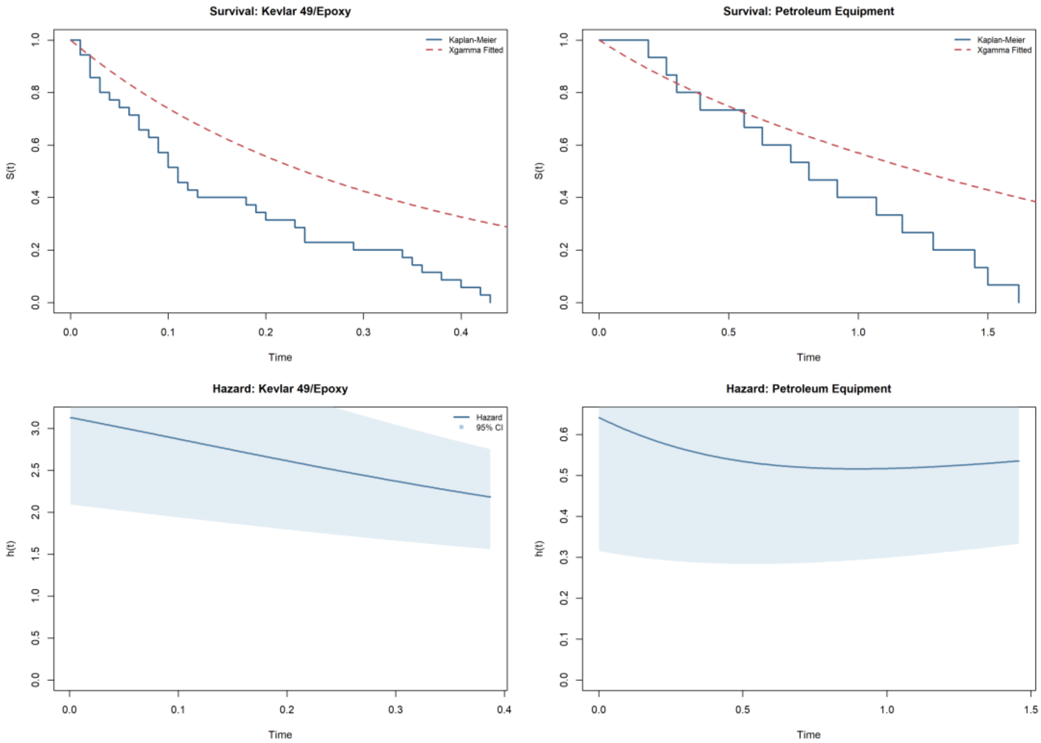

Figure 7: Estimated Survival Curves (MLE And Bayesian) Overlaid with the Kaplan–Meier Estimator for Both Datasets.

Figure 8 presents additional goodness-of-fit diagnostic plots, including PP-plots and QQ-plots.

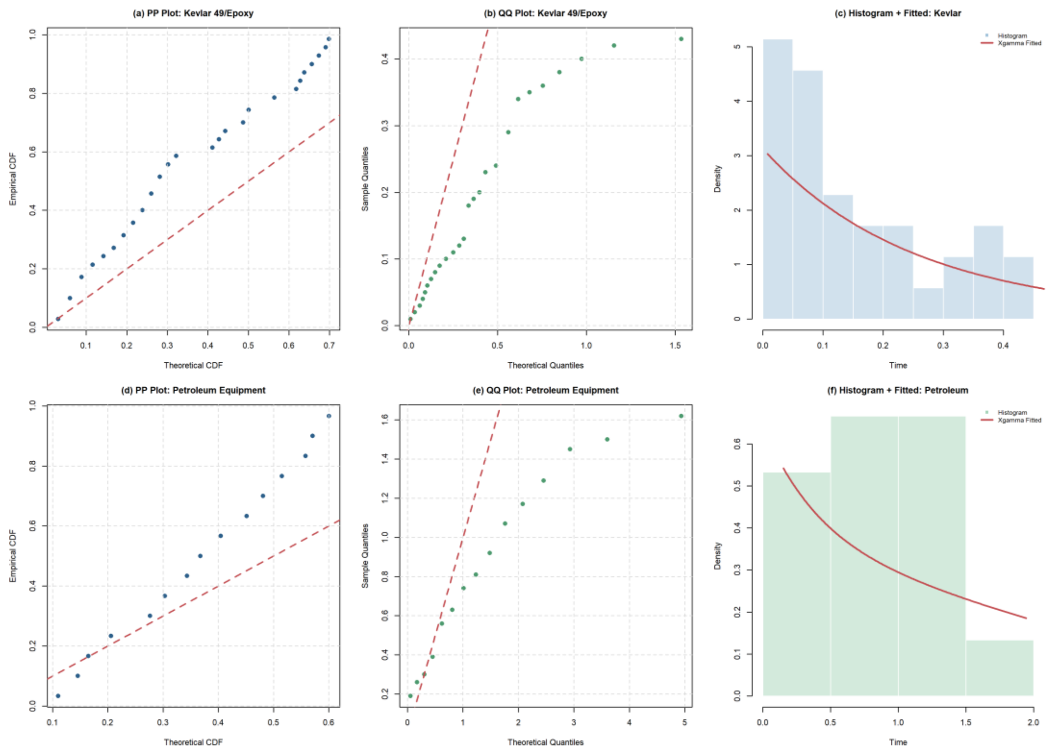

Figure 8: Goodness-Of-Fit Diagnostic Plots: PP-Plots And QQ-Plots for Both Datasets.

Figure 9 compares the estimated hazard functions from the MLE and Bayesian approaches.

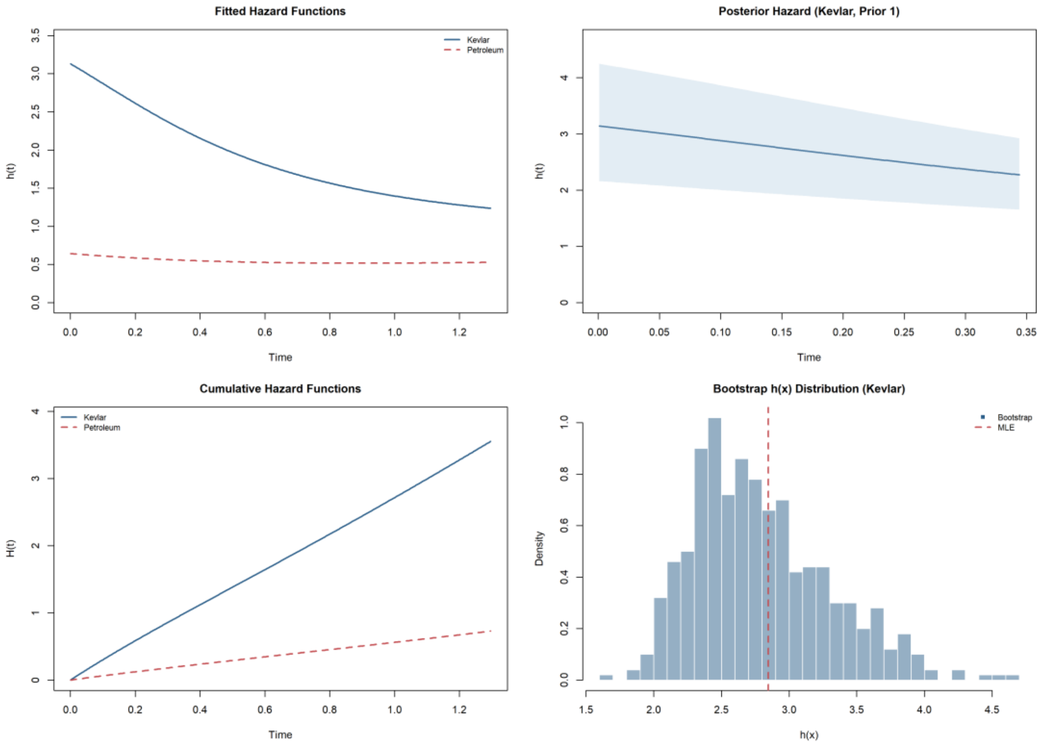

Figure 9: Comparison Of Estimated Hazard Functions from MLE And Bayesian Estimation for Both Datasets.

Figure 10 provides an overall summary comparison of all estimation methods across both datasets.

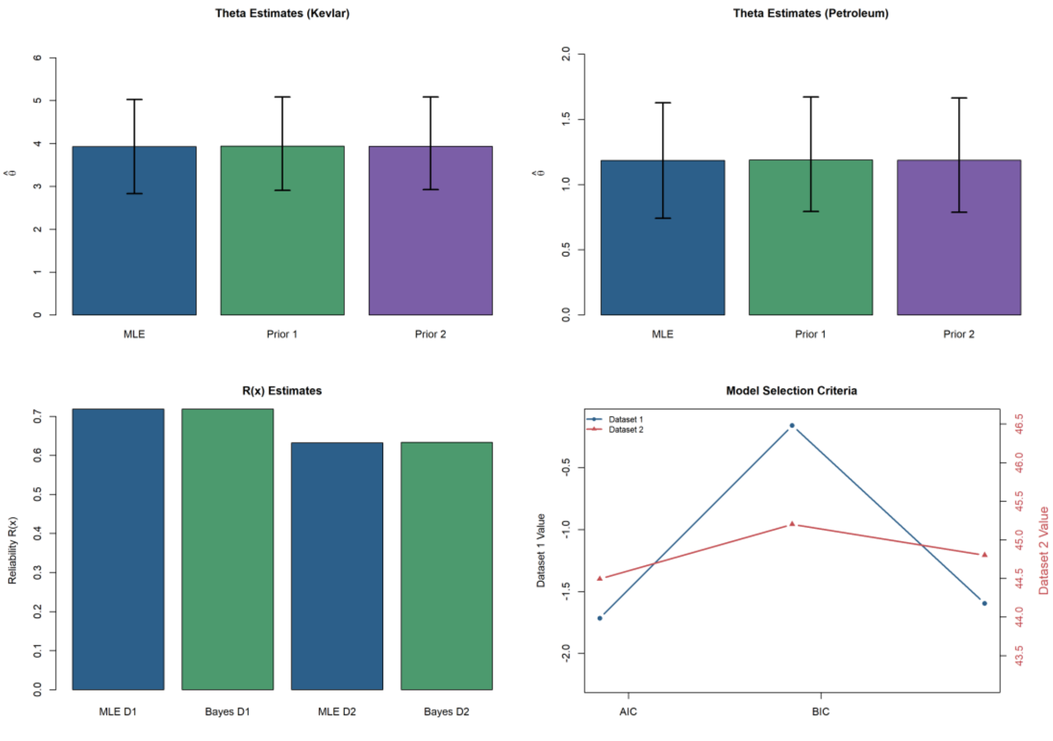

Figure 10: Summary Comparison Of MLE, Bayesian (Prior 1 And Prior 2), And Bootstrap Estimates Across Both Datasets.

7. CONCLUSIONS

In this article, the classical and Bayesian inference procedures of the Xgamma distribution under unified hybrid censoring are developed, which is a novel contribution. For classical inference, maximum likelihood estimation is carried out through BFGS optimization with log-transformation. Four types of confidence intervals (ACI-NA, ACI-NL, percentile bootstrap, bootstrap-t) are constructed. For Bayesian inference, MCMC simulation with Robbins–Monro adaptive proposal is used with two different gamma prior distributions. These inference procedures are systematically developed and compared.

In this simulation study, there are a total of nine design plans, which are grouped into three design groups. For each design group, there are 200 replications. From this simulation, several interesting results are obtained. First, the Bayesian estimator with data-adaptive Prior 2 has the minimum RMSE over all plans, although the improvement over the MLE is not dramatic, with an improvement of 1–2%. Second, the RMSE of the estimator $\hat{\theta}$ decreases by 11% when the sample size is increased from $n = 20$ to $n = 50$, which is as expected because the estimator is consistent. Third, over the four intervals, the best coverage probability is obtained with the ACI-NL interval, with coverage probabilities ranging from 0.510 to 0.750, and the shortest interval length is obtained with the HPD interval. Fourth, as the level of censoring increases, the estimation uncertainty also increases, but the interval becomes wider and more reliable.

The applicability of the proposed methods to two real data sets — one concerning the failure of Kevlar 49/epoxy strand and the other the failure of equipment in a petroleum refinery — is demonstrated. In both data sets, the maximum likelihood estimates and the Bayesian estimates have a relative difference of 0.35% or less for all parameters, indicating that the data are informative enough to render the distinction between the two methods negligible. In both data sets, the goodness-of-fit test rejects the Xgamma distribution at $\alpha = 0.05$, owing to the lack of flexibility of a one-parameter distribution.

There are several avenues of future research that have been highlighted by the current study. Extensions of the proposed methods to multi-parameter generalizations of the Xgamma distribution, such as the two-parameter generalized Xgamma distribution, under UHC would be beneficial. Another important area of future research would be the determination of the optimum censoring scheme, including the choice of r, T_1, and T_2 that would lead to the minimization of the expected variance or the maximization of the Fisher information [34]. Extensions of the proposed methods to accommodate the case of competing risks [7, 14] and progressive censoring schemes [8, 35, 3] would be important areas of future research. Extensions of the proposed methods to accommodate the case of joint inference of several samples under UHC [36, 37, 38] and accelerated life test problems [39, 40, 41, 42] would also be important areas of future research.

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