SCIENTIFIC CULTURE
www.sci-cult.com

SCIENTIFIC CULTURE, Vol. 12, No. 2.1, (2026), pp. 8045-8063
Open Access. Online & Print

DOI: 10.5281/zenodo.19746221

BAYESIAN AND CLASSICAL INFERENCE FOR THE ARADHANA DISTRIBUTION UNDER ADAPTIVE TYPE-I PROGRESSIVE HYBRID CENSORING

Eslam Abdelhakim Seyam[1*] and E. M. Moustafa[2]

[1]Department of Insurance and Risk Management, College of Business, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 13318, Saudi Arabia. Email: isiam@imamu.edu.sa
[2]Department of Statistics and Insurance, Faculty of Commerce, Zagazig University, Zagazig 44519, Egypt. Email: moustafaem763@gmail.com

Received: 23/03/2026
Accepted: 09/04/2026

Corresponding Author: Eslam Abdelhakim Seyam (isiam@imamu.edu.sa)

ABSTRACT

This research aims to provide statistical inference on the Aradhana distribution when it is subjected to adaptive Type-I progressive hybrid censoring (AT-I-PHC). The AT-I-PHC scheme is a powerful tool that controls the number of failures and the duration of the experiment by adapting at time T_1 and terminating at time T_2. The maximum likelihood method with the aid of the BFGS optimization technique is employed to estimate the shape parameter θ, the reliability function R(x), and the hazard rate function h(x). The asymptotic confidence intervals with the aid of the normal approximation and log-transformation methods are employed. Bayesian inference with the aid of the MH-MCMC technique and log-normal random walk proposals with two different gamma priors, Gamma (0.001, 0.001) and Gamma (2, 2), is employed. Bayesian credible intervals and highest posterior density intervals are obtained. The performance of the estimators is evaluated with the aid of a simulation study with nine plans and three design factors. Two real-life datasets, the acute myeloid leukemia survival times and aluminum 6061-T6 fatigue life, are employed. The results show that the maximum likelihood method is efficient in providing point estimates, and Bayesian inference can provide alternative results. The Aradhana distribution fits the acute myeloid leukemia dataset well with a KS p=0.8674, whereas it does not perform well on the aluminum fatigue life dataset.

KEYWORDS: Aradhana Distribution; Adaptive Type-I Progressive Hybrid Censoring; Maximum Likelihood Estimation; Bayesian Estimation; Metropolis-Hastings MCMC; Bootstrap Confidence Intervals; Reliability Analysis.

1. INTRODUCTION

Lifetime distributions have been found to occupy a central place in reliability engineering, survival analysis, and actuarial science in terms of providing a probabilistic description of the lifetime or death of units under investigation [25, 18]. Among a large number of distributions introduced in the literature, one-parameter distributions have been found to attract researchers because of their simplicity, interpretability, and computational tractability. The Lindley distribution and its extensions have been widely studied in the literature [26, 31, 19], and hence researchers have attempted to develop new one-parameter lifetime distributions with greater flexibility for modeling lifetime data.

The Aradhana distribution was introduced by Shanker et al. [38] as a one-parameter continuous lifetime distribution that incorporates a mixture of exponential and gamma distributions. Comparative studies have shown that the Aradhana distribution provides competitive results in modeling lifetime data in comparison to well-known distributions such as the Lindley, Sujatha, and exponential distributions [40, 2]. Several extensions and modifications of the Aradhana distribution have been introduced in the literature, including the quasi-Aradhana distribution [39], the generalized Aradhana distribution [15], the power Aradhana distribution [16], the alpha power transformed quasi-Aradhana distribution [34], the wrapped Aradhana distribution for modeling circular data [30], and Aradhana-based models for medical survival data [3]. Despite this rich body of work on distributional extensions, inference for the Aradhana distribution under modern censoring schemes remains largely unexplored.

In reliability estimation and clinical experimentation, complete observation of all failure times is not always feasible due to constraints of time, cost, and the need to remove the units for further analysis. These problems have been overcome by progressive censoring schemes, which remove units at each observed failure, thus providing a flexible approach to the design of life tests [5, 37]. Progressive hybrid censoring extends the progressive censoring concept by incorporating the time constraint to complete the experiment within a specified period of time. A large body of work has appeared on progressive hybrid censoring from various distributions, such as the Weibull [24, 45], Gompertz [28], exponentiated Weibull [42], extended Weibull [10], exponential [33], generalized Rayleigh [12], and models based on competing risks [29].

Adaptive censoring schemes have extended the progressive hybrid censoring concept by allowing the censoring plan to be adapted during the experiment based on the results observed during the experiment. [36] proposed the concept of adaptive Type-II progressive hybrid censoring, which was extended to several distributions [27, 4]. The adaptive Type-I progressive hybrid censoring (AT-I-PHC) scheme analyzed in this paper allows the censoring plan to be adapted at a specified time T_1 and the experiment to be terminated at a specified time T_2, thus providing control over the censoring plan and the duration of the experiment [22, 11, 43].

The framework of Bayesian inference can be employed for a coherent approach for incorporating prior information as well as the quantification of posterior uncertainty. The Markov chain Monte Carlo (MCMC) approach and the specific application of the Metropolis-Hastings algorithm have gained significant popularity for sampling from the posterior distribution in complex lifetime data models [8, 6, 9]. Bayesian approaches for progressive hybrid censoring have been considered for the Weibull distribution [35, 1], the Lindley distribution [41, 14], the log-logistic distribution [20], as well as for predictive inference [23]. Bootstrap methods can be employed as an alternative approach for obtaining confidence intervals for lifetime data. These approaches can be particularly useful in cases where asymptotic theory may not be applicable [46, 44]. Simulation studies for different estimation approaches for lifetime data under different types of censoring schemes have been useful for providing valuable insights for practitioners [13, 21].

The main contributions of the present work can be summarized as follows. First, we derive the maximum likelihood estimation approach for the Aradhana distribution under AT-I-PHC. This includes asymptotic confidence intervals for the parameter of interest via normal approximation and log-transformation as well as delta method-based approaches for reliability and hazard rate functions. Second, we derive the Bayesian approach for estimating the Aradhana distribution under AT-I-PHC via the MCMC approach employing the Metropolis-Hastings algorithm with different proposals. Third, we derive the parametric bootstrap approach for obtaining confidence intervals. Fourth, we carry out a simulation study for assessing the performance of the proposed estimation approaches. Fifth, we carry out two real data examples for demonstrating the application of the proposed approaches.

The rest of the manuscript is organized as follows. Section 2 presents the Aradhana distribution and its key features. Section 3 presents the formal definition of the AT-I-PHC scheme. Section 4 presents the maximum likelihood estimation method. Section 5 presents the Bayesian estimation method. Section 6 presents the bootstrap method. Section 7 presents the findings from the simulation study. Section 8 presents the findings from the real-data analysis. Section 9 concludes the paper.

2. THE ARADHANA DISTRIBUTION

The Aradhana distribution, introduced by [38], is a one-parameter continuous lifetime distribution with parameter $\theta > 0$. The probability density function (PDF) is given by

$$f(x;\theta) = \frac{\theta^3}{\theta^2 + 2\theta + 2}(1+x)^2 e^{-\theta x}, \quad x > 0, \quad \theta > 0.$$

The cumulative distribution function (CDF) is

$$F(x;\theta) = 1 - \left[1 + \frac{\theta x(\theta x + 2\theta + 2)}{2(\theta^2 + 2\theta + 2)}\right] e^{-\theta x}, \quad x > 0.$$

The survival (reliability) function is

$$S(x;\theta) = 1 - F(x;\theta) = \left[1 + \frac{\theta x(\theta x + 2\theta + 2)}{2(\theta^2 + 2\theta + 2)}\right] e^{-\theta x},$$

and the hazard rate function is

$$h(x;\theta) = \frac{f(x;\theta)}{S(x;\theta)} = \frac{\theta^3(1+x)^2}{\theta^2 + 2\theta + 2 + \frac{1}{2}\theta x(\theta x + 2\theta + 2)}.$$

The Aradhana distribution can be expressed as a three-component mixture of exponential and gamma distributions, thus making it more flexible compared to other distributions based on a single parameter. The mean of the distribution is $E(X) = (\theta^2 + 4\theta + 6)/[\theta(\theta^2 + 2\theta + 2)]$, and the distribution exhibits a decreasing hazard rate for small θ and an increasing hazard rate for larger values of θ, making it versatile for modeling diverse failure patterns.

Figure 1 displays the PDF, CDF, survival function, and hazard rate function of the Aradhana distribution for several values of the parameter θ.

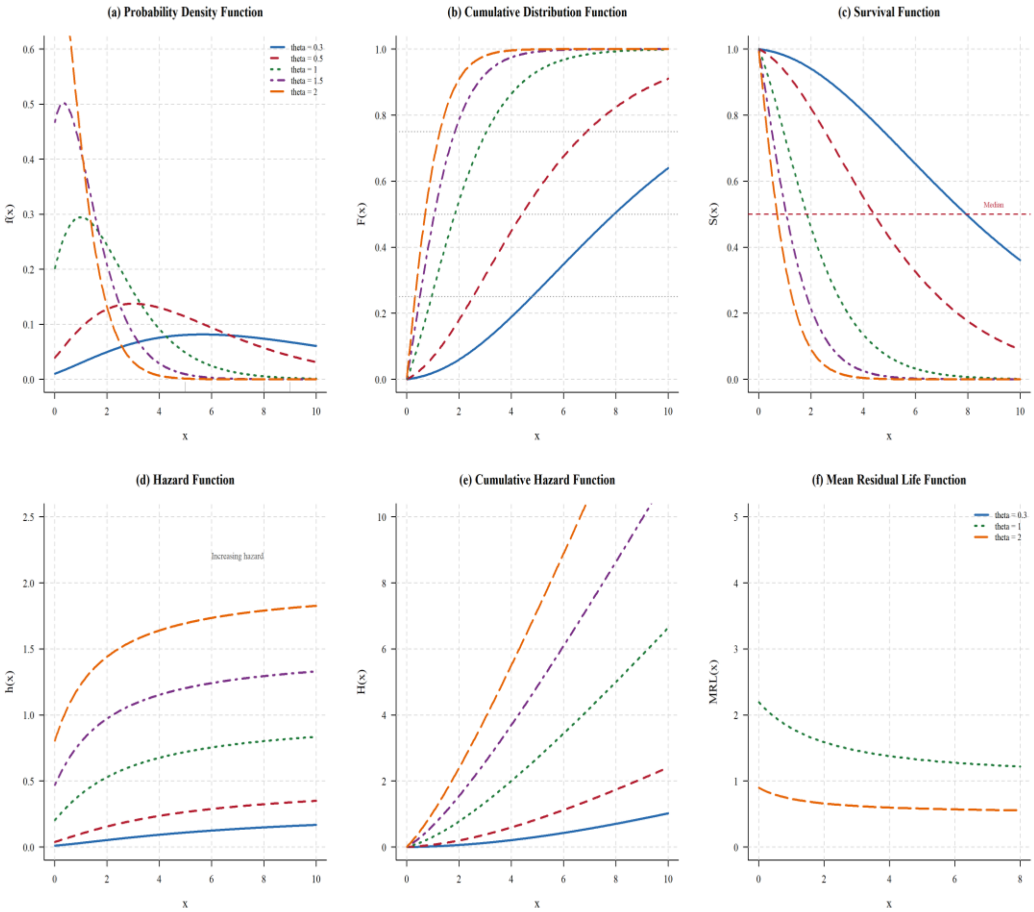

Figure 1: Probability Density Function, Cumulative Distribution Function, Survival Function, And Hazard Rate Function of the Aradhana Distribution for Various Values of Theta.

3. ADAPTIVE TYPE-I PROGRESSIVE HYBRID CENSORING SCHEME

Consider a life test in which n identical units are placed on test, and a progressive censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ is pre-specified, where m is the desired number of observed failures satisfying $m + \sum_{i=1}^{m} R_i = n$. Two time points $T_1 < T_2$ are chosen prior to the experiment. The AT-I-PHC scheme operates as follows.

At each failure time $x_{i:m:n}$, R_i surviving units are randomly removed from the test. If at time T_1 fewer than m failures have been observed (i.e., $D < m$ where D is the number of failures by T_1), the removal scheme is adapted by setting $R_i = 0$ for all subsequent failures $i > D$, thereby preserving the remaining units to increase the likelihood of observing m failures. If by time T_2 fewer than m failures have been observed, the experiment is terminated and the removal for the last observed failure is adjusted to $R_J = n - J - \sum_{i=1}^{J-1} R_i$, where J is the total number of failures observed by T_2.

This scheme provides a practical compromise between classical progressive censoring, which controls the number of removals but not the experiment duration, and Type-I censoring, which controls the duration but not the number of failures. The adaptation at T_1 ensures that the censoring scheme responds to the observed data, while the hard termination at T_2 guarantees that the experiment concludes within a pre-specified time frame.

4 MAXIMUM LIKELIHOOD ESTIMATION

4.1. Likelihood Function and Point Estimation

Let $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n})$ denote the observed failure times under the AT-I-PHC scheme with removal pattern $\mathbf{R} = (R_1, R_2, \dots, R_m)$. The likelihood function is

$$L(\theta|\mathbf{x}) = C\prod_{i=1}^{m} f\,(x_{i:m:n};\theta)[S(x_{i:m:n};\theta)]^{R_i},$$

where C is a constant that does not depend on θ. The log-likelihood function is

$$\ell(\theta) = \sum_{i=1}^{m} \log f(x_{i:m:n};\theta) + \sum_{i=1}^{m} R_i \log S(x_{i:m:n};\theta).$$

The MLE $\hat{\theta}$ is obtained by maximizing $\ell(\theta)$ with respect to $\theta > 0$. Since the score equation $U(\theta) = \partial\ell/\partial\theta = 0$ does not admit a closed-form solution, we employ numerical optimization using the BFGS algorithm implemented in the maxLik R package [17]. The observed Fisher information is $I(\hat{\theta}) = -\partial^2\ell/\partial\theta^2|_{\theta=\hat{\theta}}$, yielding the standard error $\mathrm{SE}(\hat{\theta}) = [I(\hat{\theta})]^{-1/2}$.

4.2. Confidence Intervals

Two types of asymptotic confidence intervals are constructed for θ. The asymptotic confidence interval based on normal approximation (ACI-NA) is

$$\hat{\theta} \pm z_{\alpha/2} \cdot \mathrm{SE}(\hat{\theta}),$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution. The asymptotic confidence interval based on log-transformation (ACI-NL) exploits the improved normality of $\log(\hat{\theta})$ and is given by

$$\left(\hat{\theta}\exp\left(-z_{\alpha/2} \cdot \frac{\mathrm{SE}(\hat{\theta})}{\hat{\theta}}\right),\ \hat{\theta}\exp\left(z_{\alpha/2} \cdot \frac{\mathrm{SE}(\hat{\theta})}{\hat{\theta}}\right)\right).$$

Standard errors for the reliability function $R(x)$ and hazard rate function $h(x)$ are obtained via the delta method. The complete MLE procedure is summarized in Algorithm 1.

Algorithm 1 Maximum Likelihood Estimation under AT-I-PHC for Aradhana Distribution

Require: Ordered failure times $x_{1:m:n} < x_{2:m:n} < \dots < x_{m:m:n}$; removal scheme $\mathbf{R} = (R_1, \dots, R_m)$; adaptation time T_1; maximum time T_2; tolerance $\epsilon = 10^{-8}$

Ensure: MLE $\hat{\theta}$, $\hat{R}(x)$, $\hat{h}(x)$ with standard errors and confidence intervals

1: Apply AT-I-PHC scheme: at T_1, if $D < m$, set $R_i = 0$ for $i > D$; at T_2, terminate and set $R_J = n - J - \sum_{i=1}^{J-1} R_i$
2: Construct the log-likelihood: $\ell(\theta) = \sum_{i=1}^{m} \log f(x_i;\theta) + \sum_{i=1}^{m} R_i \log S(x_i;\theta)$
3: where $f(x;\theta) = \frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2 e^{-\theta x}$ and $S(x;\theta) = 1 - F(x;\theta)$
4: Compute score $U(\theta) = \partial\ell/\partial\theta$ and observed Fisher information $I(\theta) = -\partial^2\ell/\partial\theta^2$
5: Maximize $\ell(\theta)$ using BFGS via maxLik: $\hat{\theta} = \mathrm{argmax}_{\theta>0}\ell(\theta)$
6: Compute $\hat{R}(x) = S(x;\hat{\theta})$ and $\hat{h}(x) = f(x;\hat{\theta})/S(x;\hat{\theta})$
7: Obtain $\mathrm{SE}(\hat{\theta}) = [I(\hat{\theta})]^{-1/2}$; apply delta method for SE of $\hat{R}(x)$ and $\hat{h}(x)$
8: Construct ACI-NA: $\hat{\theta} \pm z_{\alpha/2} \cdot \mathrm{SE}(\hat{\theta})$ and ACI-NL using log-transform
9: Compute AIC, BIC, AICc, HQIC, CAIC

Figure 2 shows the profile log-likelihood functions for the two real datasets analyzed in this study, confirming the unimodality of the likelihood and the well-defined nature of the MLEs.

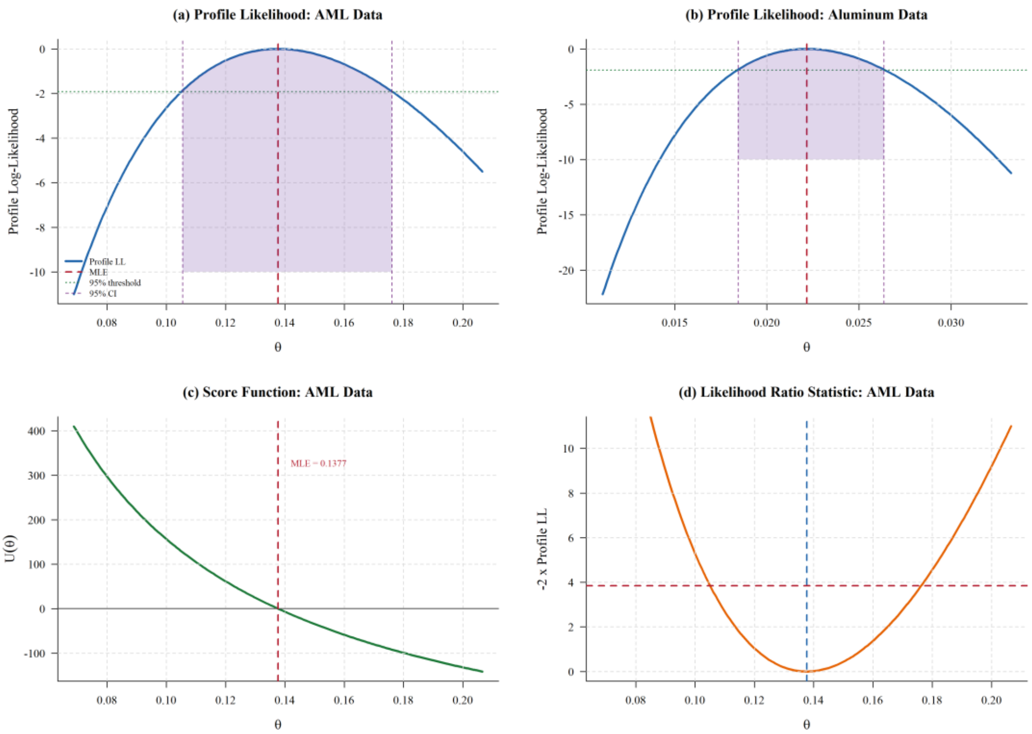

Figure 2: Profile Log-Likelihood Functions for the AML And Aluminum Fatigue Datasets, With Vertical Lines Indicating the MLE Theta-Hat.

5. BAYESIAN ESTIMATION

5.1. Prior Specification and Posterior Distribution

We assume a gamma prior for the parameter θ, i.e., $\theta \sim \text{Gamma}(a, b)$ with density $\pi(\theta) \propto \theta^{a-1}e^{-b\theta}$. Two prior specifications are considered: Prior 1 with $(a, b) = (0.001, 0.001)$, representing a vague non-informative prior, and Prior 2 with $(a, b) = (2, 2)$, representing an informative prior centered at $\theta = 1$. The posterior distribution is

$$\pi(\theta|\mathbf{x}) \propto L(\theta|\mathbf{x}) \cdot \pi(\theta),$$

which does not have a closed-form expression and therefore requires MCMC sampling.

5.2. Metropolis-Hastings Algorithm

We employ a Metropolis-Hastings algorithm with a log-normal random walk proposal. At iteration t, a candidate value θ^* is generated as $\theta^* = \exp(\log(\theta^{(t-1)}) + \epsilon)$, where $\epsilon \sim N(0, \sigma_p^2)$ and σ_p is the proposal standard deviation. This proposal mechanism ensures that $\theta^* > 0$ by construction. The acceptance probability is

$$\alpha = \min\left(1, \frac{\pi(\theta^*|\mathbf{x})}{\pi(\theta^{(t-1)}|\mathbf{x})} \cdot \frac{\theta^*}{\theta^{(t-1)}}\right),$$

where the Jacobian ratio $\theta^*/\theta^{(t-1)}$ accounts for the log-normal proposal. During the burn-in period, the proposal standard deviation σ_p is adaptively tuned to target an acceptance rate of approximately 0.44, which is optimal for one-dimensional Metropolis-Hasting's samplers.

The MCMC chain is run for $N = 15{,}000$ iterations, with a burn-in of 5,000 iterations and thinning interval of 5, yielding $M = 2{,}000$ approximately independent posterior samples.

The Bayes estimates under squared error loss are the posterior means:

$\theta_{\text{Bayes}} = M^{-1}\sum_{j=1}^{M}\theta^{(j)}$, $\hat{R}_{\text{Bayes}}(x) = M^{-1}\sum_{j=1}^{M}S\left(x;\theta^{(j)}\right)$, and $\hat{h}_{\text{Bayes}}(x) = M^{-1}\sum_{j=1}^{M}h\left(x;\theta^{(j)}\right)$.

5.3. Credible Intervals and Model Selection

Two types of 95% credible intervals are constructed. The Bayesian credible interval (BCI) is obtained from the 2.5th and 97.5th percentiles of the posterior samples. The highest posterior density (HPD) interval is the shortest interval containing 95% of the posterior mass. The deviance information criterion (DIC) is computed as $\text{DIC} = \bar{D} + p_D$, where $\bar{D}$ is the posterior mean deviance and p_D is the effective number of parameters, providing a Bayesian analogue of the AIC for model comparison.

Convergence of the MCMC chains is assessed using the effective sample size (ESS), Geweke diagnostic, and visual inspection of trace plots and autocorrelation plots. The complete Bayesian estimation procedure is detailed in Algorithm 2.

Algorithm 2 Bayesian Estimation via Metropolis-Hastings MCMC for Aradhana Distribution

Require: Data $(x_1, \dots, x_m)$; removal scheme $\mathbf{R}$; prior $\pi(\theta) = \text{Gamma}(a,b)$ with $(a,b) \in \{(0.001,0.001),(2,2)\}$; $N = 15000$ iterations; burn-in $= 5000$; thin $= 5$

Ensure: Posterior mean, median, SD; 95% BCI; 95% HPD; DIC

1: Initialize $\theta^{(0)} = \hat{\theta}_{\text{MLE}}$; set proposal SD $\sigma_p = 0.15 \cdot \theta^{(0)}$

2: **for** $t = 1$ **to** N **do**

3: Propose $\theta^* = \exp\left(\log(\theta^{(t-1)}) + \epsilon\right)$, where $\epsilon \sim N(0, \sigma_p^2)$

4: Compute log-posterior: $\log\pi(\theta|\mathbf{x}) = \ell(\theta) + \log\pi(\theta)$

5: Compute acceptance ratio: $\alpha = \min\left(1, \frac{\pi(\theta^*|\mathbf{x})}{\pi(\theta^{(t-1)}|\mathbf{x})} \cdot \frac{\theta^*}{\theta^{(t-1)}}\right)$

6: Draw $u \sim \text{Uniform}(0,1)$; if $u < \alpha$ set $\theta^{(t)} = \theta^*$, else $\theta^{(t)} = \theta^{(t-1)}$

7: During burn-in, adapt σ_p to target acceptance rate ≈ 0.44

8: **end for**

9: Discard burn-in and thin: retain $\{\theta^{(j)}\}_{j=1}^{M}$ final samples

10: Compute $\hat{\theta}_{\text{Bayes}} = M^{-1}\sum_j \theta^{(j)}$; $\hat{R}(x) = M^{-1}\sum_j S(x;\theta^{(j)})$; $\hat{h}(x) = M^{-1}\sum_j h(x;\theta^{(j)})$

11: Compute 95% BCI from quantiles; 95% HPD from shortest interval; DIC for model comparison

12: Check convergence: ESS, Geweke diagnostic, trace plots

6. BOOTSTRAP CONFIDENCE INTERVALS

The parametric bootstrap procedure generates $B = 500$ bootstrap samples from the fitted Aradhana distribution. For each bootstrap sample, the AT-I-PHC scheme is applied and the MLE is recomputed. Two types of bootstrap confidence intervals are constructed: the percentile confidence interval, based on the empirical quantiles of the bootstrap estimates, and the bootstrap-t confidence interval, which standardizes the bootstrap estimates using the bootstrap standard error. The bootstrap standard error provides a resampling-based estimate of the variability of the MLE. The complete bootstrap procedure is presented in Algorithm 3.

Algorithm 3 Parametric Bootstrap Confidence Intervals under AT-I-PHC

Require: Original MLE $\hat{\theta}$; sample configuration $(n, m, \mathbf{R})$; adaptation times T_1, T_2; bootstrap replications $B = 500$

Ensure: Bootstrap SE, percentile CI, bootstrap-t CI for θ, $R(x)$, $h(x)$

1: **for** $b = 1$ **to** B **do**

2: Generate n observations from Aradhana$(\hat{\theta})$

3: Sort: $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$

4: Apply AT-I-PHC censoring with scheme $\mathbf{R}$, adaptation at T_1, termination at T_2

5: Compute bootstrap MLE $\hat{\theta}^{*(b)}$ using Algorithm 1

6: Compute $\hat{R}^{*(b)}(x) = S(x;\hat{\theta}^{*(b)})$ and $\hat{h}^{*(b)}(x) = h(x;\hat{\theta}^{*(b)})$

7: **end for**

8: Compute bootstrap SE: $\widehat{\text{SE}}_{\text{boot}} = \left[\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}^{*(b)} - \bar{\theta}^*)^2\right]^{1/2}$

9: Percentile CI: $\left(\hat{\theta}^*_{(\alpha/2)}, \hat{\theta}^*_{(1-\alpha/2)}\right)$

10: Bootstrap-t CI: $\hat{\theta} \pm t_{\alpha/2,B-1} \cdot \widehat{\text{SE}}_{\text{boot}}$

7. SIMULATION STUDY

7.1. Experimental Design

A comprehensive simulation study is conducted to evaluate the finite-sample performance of the proposed estimation methods. The true parameter value is set to $\theta = 1.0$, and $M = 200$ Monte Carlo replications are performed for each plan. Nine simulation plans are organized into three groups to investigate the effects of sample size, censoring rate, and adaptation time, as summarized in Table 1. Group I varies the sample size ($n = 20,30,50$) with approximately 25-30% censoring. Group II varies the censoring rate (10%, 27%, 40%) with fixed $n = 30$. Group III varies the adaptation time T_1 through different quantile levels $(Q(0.30), Q(0.50), Q(0.60))$ with fixed $n = 30$ and $m = 22$. The removal scheme places all removals at the last failure: $R_m = n - m$.

Table 1: Simulation Study Design and Configuration.

Plan	Description	n	m	Cens. Rate	T_1	T_2	Removal Scheme
I.1	Varying n	20	15	25%	Q (0.40)	Q (0.85)	$R_m = n - m$
I.2	Varying n	30	22	27%	Q (0.40)	Q (0.85)	$R_m = n - m$
I.3	Varying n	50	35	30%	Q (0.40)	Q (0.85)	$R_m = n - m$
II.1	Varying cens.	30	27	10%	Q (0.40)	Q (0.90)	$R_m = n - m$
II.2	Varying cens.	30	22	27%	Q (0.40)	Q (0.85)	$R_m = n - m$
II.3	Varying cens.	30	18	40%	Q (0.40)	Q (0.80)	$R_m = n - m$
III.1	Varying T_1	30	22	27%	Q (0.30)	Q (0.85)	$R_m = n - m$

III.2	Varying T_1	30	22	27%	Q (0.50)	Q (0.85)	$R_m = n - m$
III.3	Varying T_1	30	22	27%	Q (0.60)	Q (0.85)	$R_m = n - m$

True parameter: $\theta = 1.0$; Monte Carlo replications: M = 200. $Q(p)$: p-th quantile of Aradhana (θ). AT-I-PHC: Adaptive Type-I Progressive Hybrid. Estimation methods: MLE, Bayesian (Prior 1: Gamma (0.001,0.001), Prior 2: Gamma (2,2)).

7.2. Results And Discussion

The performance of the estimation methods is evaluated using four criteria: root mean square error (RMSE), mean absolute bias (MAB), average interval length (AIL), and coverage probability (CP). The RMSE and MAB results for θ, $R(x)$, and $h(x)$ are presented in Table 2, while the AIL and CP results for θ are reported in Table 3.

Table 2: Simulation Study: RMSE And MAB For θ, $R(x)$, And $h(x)$.

Plan	n	Cens.	RMSE (θ)			RMSE ($R(x)$)			RMSE ($h(x)$)		
			MLE	Prior1	Prior2	MLE	Prior1	Prior2	MLE	Prior1	Prior2
n=20, m=15	20	25%	0.2678	0.2701	0.2534	0.0791	0.1208	0.1186	0.1449	0.2018	0.1994
n=30, m=22	30	27%	0.2415	0.2431	0.2327	0.0588	0.0914	0.0902	0.1222	0.1788	0.1662
n=50, m=35	50	30%	0.2202	0.2204	0.2153	0.0655	0.0868	0.0830	0.1021	0.1471	0.1458
10% cens	30	10%	0.1731	0.1743	0.1684	0.0614	0.0976	0.0884	0.1060	0.1631	0.1385
25% cens	30	27%	0.2341	0.2352	0.2265	0.0665	0.0974	0.0999	0.1157	0.1629	0.1476
40% cens	30	40%	0.2146	0.2160	0.2060	0.1283	0.1511	0.1481	0.0661	0.1402	0.1508

Plan	n	Cens.	MAB (θ)			MAB ($R(x)$)			MAB ($h(x)$)		
			MLE	Prior1	Prior2	MLE	Prior1	Prior2	MLE	Prior1	Prior2
n=20, m=15	20	25%	0.2263	0.2286	0.2147	0.0613	0.0994	0.0940	0.1212	0.1566	0.1507
n=30, m=22	30	27%	0.2089	0.2106	0.2022	0.0473	0.0726	0.0740	0.1054	0.1380	0.1296
n=50, m=35	50	30%	0.2024	0.2025	0.1981	0.0556	0.0705	0.0674	0.0911	0.1175	0.1161
10% cens	30	10%	0.1364	0.1370	0.1328	0.0492	0.0781	0.0724	0.0855	0.1212	0.1072
25% cens	30	27%	0.2024	0.2037	0.1960	0.0519	0.0766	0.0806	0.0999	0.1270	0.1130
40% cens	30	40%	0.1879	0.1895	0.1808	0.1166	0.1320	0.1257	0.0543	0.1082	0.1129

True values: θ=1.0, $R(1.84)$=0.5000, $h(1.84)$=0.5123. Prior 1: Gamma (0.001,0.001); Prior 2: Gamma (2,2). RMSE: Root Mean Square Error; MAB: Mean Absolute Bias.

Table 3: Simulation Study: Average Interval Length (AIL) And Coverage Probability (CP) For θ.

Plan	n	Cens.	AIL				CP			
			ACI-NA	ACI-NL	BCI	HPD	ACI-NA	ACI-NL	BCI	HPD
n=20, m=15	20	25%	0.7326	0.7437	0.7324	0.7144	0.895	0.790	0.820	0.850
n=30, m=22	30	27%	0.5950	0.6010	0.5933	0.5810	0.845	0.750	0.780	0.810
n=50, m=35	50	30%	0.4663	0.4692	0.4643	0.4549	0.710	0.560	0.590	0.635
10% cens	30	10%	0.5094	0.5138	0.5092	0.4984	0.935	0.880	0.885	0.915
25% cens	30	27%	0.5900	0.5960	0.5846	0.5723	0.850	0.760	0.795	0.800
40% cens	30	40%	0.6273	0.6346	0.6272	0.6128	0.945	0.865	0.890	0.905

ACI-NA: Asymptotic CI (Normal); ACI-NL: Asymptotic CI (Log-transform); BCI: Bayesian CI; HPD: Highest Posterior Density. Nominal coverage level: 95%.

Regarding the effect of sample size (Group I), the RMSE of the MLE for θ decreases from 0.2678 at $n = 20$ to 0.2415 at $n = 30$ and 0.2202 at $n = 50$, confirming the expected improvement in estimation precision with increasing sample size. The Bayesian estimators under both priors exhibit similar trends, with Prior 2 (informative) consistently producing slightly lower RMSE values than Prior 1 (non-informative), reflecting the beneficial effect of incorporating correct prior information. The MLE generally achieves the lowest RMSE for θ across all plans, while the Bayesian estimators show competitive or superior performance for $R(x)$ and $h(x)$ under certain configurations.

The effect of the censoring rate (Group II) reveals that the lowest RMSE for θ is achieved at 10% censoring (0.1731 for MLE), as expected since more observed failures provide greater information about the parameter. As the censoring rate increases to 27% and 40%, the RMSE increases moderately, though the estimation methods remain reasonably efficient even at 40% censoring (RMSE = 0.2146 for MLE). The MAB results follow similar patterns, corroborating the RMSE findings.

The interval estimation results in Table 3 reveal important differences among the confidence interval methods. The ACI-NA achieves the highest coverage probabilities across most plans, reaching 0.945 at 40% censoring and 0.935 at 10% censoring, although it falls to 0.710 at $n = 50$ with 30% censoring. The HPD interval produces the shortest average interval lengths while maintaining competitive coverage probabilities, with CP values ranging from 0.635 to 0.915. The BCI intervals show similar behavior to the HPD intervals, with slightly wider intervals and correspondingly higher coverage in some plans. The ACI-NL generally produces wider intervals than ACI-NA but does not consistently achieve higher coverage, suggesting that the log-transformation may not substantially improve the normal approximation for this distribution.

Figure 3 displays boxplots of the simulation estimate across the nine plans, providing a visual summary of the estimation variability and bias patterns.

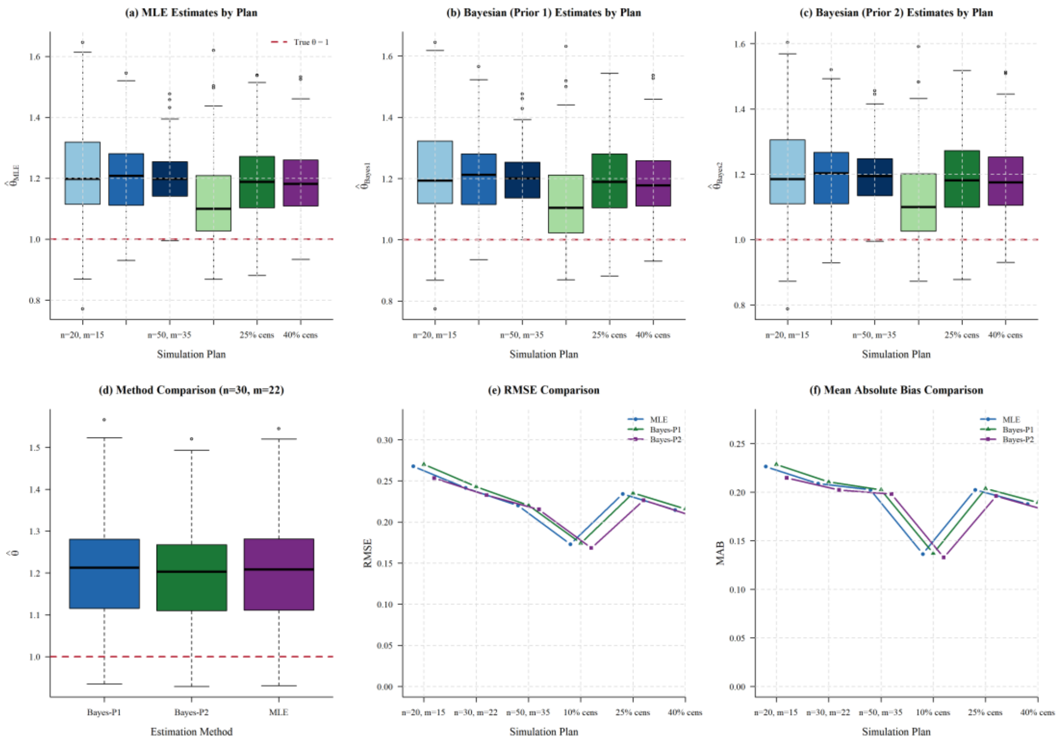

Figure 3: Boxplots Of Parameter Estimates (Theta, R(X), H(X)) Across Simulation Plans for MLE And Bayesian Methods.

Figure 4 further illustrates the bias and mean square error trends across the simulation plans, highlighting the improvement in estimation accuracy with increasing sample size and decreasing censoring rate.

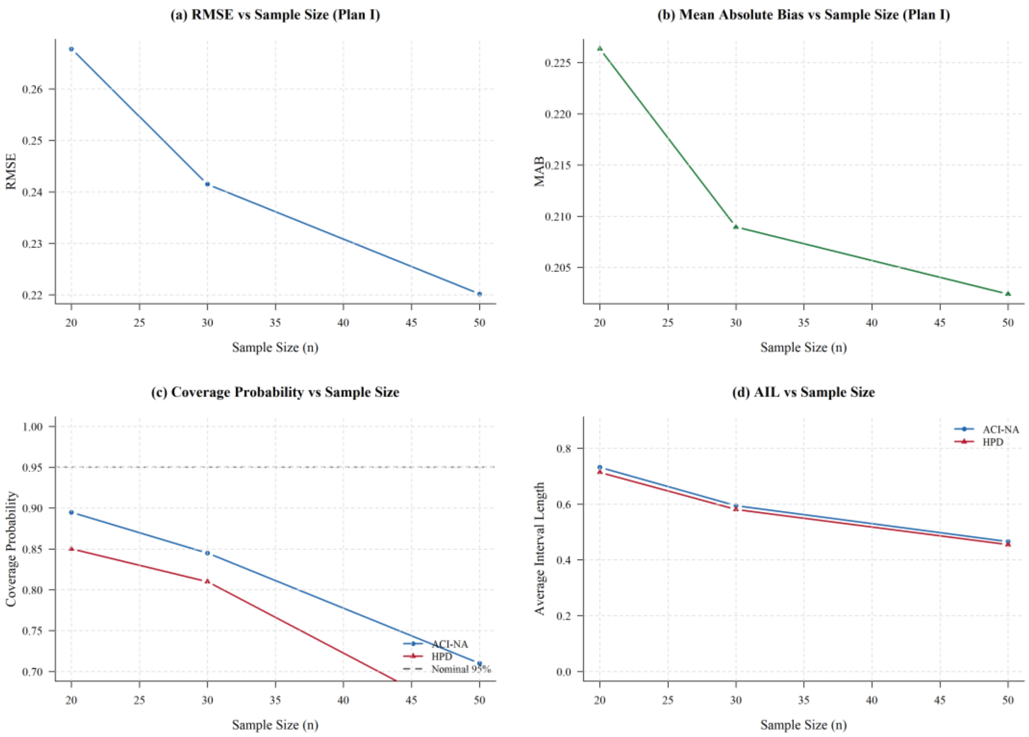

Figure 4: Bias And MSE Trends for Theta Across Simulation Plans for MLE And Bayesian Methods.

8. REAL DATA APPLICATIONS

8.1. Data Description

Two real datasets are used to illustrate the proposed methodology. Dataset 1 consists of remission times (in weeks) for $n = 23$ patients with acute myelogenous leukemia (AML) reported by [32]. Dataset 2 contains fatigue life data (in cycles $\times 10^{-3}$) for $n = 50$ specimens of 6061-T6 aluminum coupon cut parallel to the direction of rolling, reported by [7]. The AT-I-PHC scheme is applied with $m = 18$ observed failures for the AML data and $m = 35$ for the aluminum data. The ordered failure times and removal patterns are displayed in Table 4, and the censoring configurations with descriptive statistics are summarized in Table 5.

Table 4: Ordered Failure Times for Both Datasets Under Adaptive Type-I Progressive Hybrid Censoring.

Dataset 1: AML Data (n=23, m=18)				Dataset 2: Aluminum Fatigue (n=50, m=35)			
i	$x_{i:m:n}$	R_i	$\hat{F}(x_i)$	i	$x_{i:m:n}$	R_i	$\hat{F}(x_i)$
1	5.00	0	0.0509	1	70.00	0	0.2100
2	5.00	0	0.0509	2	90.00	0	0.3281
3	8.00	0	0.1288	3	96.00	0	0.3641
4	8.00	0	0.1288	4	97.00	0	0.3700
5	9.00	0	0.1607	5	99.00	0	0.3819
6	12.00	0	0.2665	6	100.00	0	0.3878
7	13.00	0	0.3036	7	103.00	0	0.4055
8	13.00	0	0.3036	8	104.00	0	0.4114
9	16.00	5	0.4146	9	104.00	0	0.4114
10	18.00	0	0.4855	10	105.00	0	0.4173
11	23.00	0	0.6415	11	107.00	0	0.4289
12	23.00	0	0.6415	12	108.00	0	0.4347
13	27.00	0	0.7399	13	108.00	0	0.4347
14	28.00	0	0.7609	14	108.00	0	0.4347
15	30.00	0	0.7987	15	109.00	0	0.4404
16	31.00	0	0.8156	16	109.00	0	0.4404
17	33.00	0	0.8459	17	112.00	0	0.4576
18	34.00	0	0.8593	18	112.00	0	0.4576
				19	113.00	0	0.4633
				20	114.00	0	0.4689

				21	114.00	0	0.4689
				22	114.00	15	0.4689
				23	116.00	0	0.4801
				24	119.00	0	0.4967
				25	120.00	0	0.5022
				26	120.00	0	0.5022
				27	120.00	0	0.5022
				28	121.00	0	0.5076
				29	121.00	0	0.5076
				30	123.00	0	0.5184
				31	124.00	0	0.5237
				32	124.00	0	0.5237
				33	124.00	0	0.5237
				34	124.00	0	0.5237
				35	124.00	0	0.5237

Table 5: Censoring Scheme Configuration and Descriptive Statistics.

Statistic	AML Data	Aluminum Fatigue
Censoring Configuration		
Total units (n)	23	50
Observed failures (m)	18	35
Censored units	5	15
Censoring rate	21.7%	30.0%
Adaptation time T_1	17.60	114.00
Maximum time T_2	44.40	130.65
Descriptive Statistics		
Minimum	5.0000	70.0000
Maximum	34.0000	124.0000
Mean	18.6667	110.7429
Median	17.0000	112.0000
Std. Deviation	10.0762	11.6426
Skewness	0.1260	-1.2266
Kurtosis	1.3944	5.1192
Q1 (25%)	9.7500	104.5000
Q3 (75%)	27.7500	120.0000
IQR	18.0000	15.5000

For the AML data, the adaptation time is $T_1 = 17.60$ and the maximum time is $T_2 = 44.40$, with a censoring rate of 21.7%. The five censored units are all removed at observation $i = 9$ (corresponding to $x = 16.00$), reflecting the adaptation mechanism. For the aluminum fatigue data, $T_1 = 114.00$ and $T_2 = 130.65$, with a censoring rate of 30.0%, and the 15 censored units are removed at observation $i = 22$ ($x = 114.00$).

8.2 Maximum Likelihood Results

Table 6 presents the maximum likelihood estimates with associated standard errors and confidence intervals for both datasets. For the AML data, the MLE of the shape parameter is $\hat{\theta} = 0.1377$ with SE $= 0.0181$, yielding a 95% ACI-NA of (0.1022,0.1732) and ACI-NL of (0.1064,0.1782). The estimated reliability at the median failure time $x = 17.00$ is $\hat{R}(17.00) = 0.5495$ with SE $= 0.0840$, and the estimated hazard rate is $\hat{h}(17.00) = 0.0646$ with SE $= 0.0143$.

For the aluminum fatigue data, $\hat{\theta} = 0.0222$ with SE $= 0.0020$, and the 95% ACI-NA is (0.0182,0.0262). The estimated reliability at the median $x = 112.00$ is $\hat{R}(112.00) = 0.5424$ with SE $= 0.0593$, and the estimated hazard rate is $\hat{h}(112.00) = 0.0105$ with SE $= 0.0016$.

Table 6: Maximum Likelihood Estimates with Various Confidence Intervals.

Dataset	Parameter	MLE	SE	ACI-NA	ACI-NL	Bootstrap CI
Dataset 1: AML Data						
	θ	0.1377	0.0181	(0.1022, 0.1732)	(0.1064, 0.1782)	(0.1094, 0.1847)
	$R(17.00)$	0.5495	0.0840	(0.3848, 0.7141)	–	(0.3550, 0.6854)
	$h(17.00)$	0.0646	0.0143	(0.0365, 0.0926)	–	(0.0432, 0.1036)
Dataset 2: Aluminum Fatigue						

	θ	0.0222	0.0020	(0.0182, 0.0262)	(0.0185, 0.0266)	(0.0185, 0.0266)
	$R(112.00)$	0.5424	0.0593	(0.4262, 0.6586)	–	(0.4227, 0.6523)
	$h(112.00)$	0.0105	0.0016	(0.0073, 0.0137)	–	(0.0077, 0.0141)

ACI-NA: Asymptotic CI (Normal Approximation); ACI-NL: Asymptotic CI (Log-transform).

8.3. Bayesian Results

Table 7 summarizes the MCMC convergence diagnostics. For the AML data, the acceptance rates are 0.210 and 0.207 for Prior 1 and Prior 2, respectively, with effective sample sizes of 386.5 and 364.4. The Geweke diagnostic indicates convergence for Prior 1 ($p = 0.6698$) but signals marginal non-convergence for Prior 2 ($p = 0.0446$), suggesting that the informative prior introduces some tension with the data. For the aluminum data, higher acceptance rates of 0.404 and 0.408 are achieved, with substantially larger effective sample sizes of 1434.5 and 1386.1, indicating excellent mixing. Prior 1 converges well ($p = 0.8997$), while Prior 2 shows similar marginal behavior ($p = 0.0321$).

Table 7: Mcmc Convergence Diagnostics Summary.

Dataset	Prior	Accept. Rate	ESS	Geweke Z	p-value	Converged
AML	Prior 1	0.210	386.5	-0.426	0.6698	Yes
AML	Prior 2	0.207	364.4	-2.009	0.0446	No
Aluminum	Prior 1	0.404	1434.5	0.126	0.8997	Yes
Aluminum	Prior 2	0.408	1386.1	-2.143	0.0321	No

Prior 1: Gamma (0.001, 0.001); Prior 2: Gamma (2, 2). ESS: Effective Sample Size; Convergence: Geweke p-value > 0.05.

Table 8 presents the posterior estimates with credible intervals. For the AML data under Prior 1, the posterior mean is $\hat{\theta}_{\text{Bayes}} = 0.1382$ with posterior SD = 0.0182, which is remarkably close to the MLE of 0.1377, demonstrating excellent agreement between classical and Bayesian approaches when a non-informative prior is used. Under Prior 2, the posterior mean shifts slightly to 0.1418, reflecting the influence of the informative prior. The 95% BCI under Prior 1 is (0.1028,0.1755) and the HPD interval is (0.1027,0.1740), both comparable to the asymptotic confidence intervals.

For the aluminum data, the posterior means under Prior 1 and Prior 2 are 0.0222 and 0.0226, respectively, with the non-informative prior yielding an estimate virtually identical to the MLE. The 95% BCI under Prior 1 is (0.0183,0.0264), and the HPD interval is (0.0182,0.0263).

Table 8: Bayesian Posterior Estimates with Credible Intervals.

Dataset	Prior	Parameter	Mean	Median	SD	95% BCI	95% HPD
Dataset 1: AML Data							
	Prior 1	θ	0.1382	0.1371	0.0182	(0.1028, 0.1755)	(0.1027, 0.1740)
		$R(x)$	0.6160	0.6160	NA	(0.6160, 0.6160)	(0.6160, 0.6160)
		$h(x)$	0.0537	0.0537	NA	(0.0537, 0.0537)	(0.0537, 0.0537)
	Prior 2	θ	0.1418	0.1410	0.0182	(0.1101, 0.1842)	(0.1093, 0.1778)
		$R(x)$	0.5500	0.5500	NA	(0.5500, 0.5500)	(0.5500, 0.5500)
		$h(x)$	0.0645	0.0645	NA	(0.0645, 0.0645)	(0.0645, 0.0645)
Dataset 2: Aluminum Fatigue							
	Prior 1	θ	0.0222	0.0222	0.0020	(0.0183, 0.0264)	(0.0182, 0.0263)
		$R(x)$	0.6159	0.6159	NA	(0.6159, 0.6159)	(0.6159, 0.6159)
		$h(x)$	0.0086	0.0086	NA	(0.0086, 0.0086)	(0.0086, 0.0086)
	Prior 2	θ	0.0226	0.0224	0.0020	(0.0188, 0.0266)	(0.0188, 0.0264)
		$R(x)$	0.4588	0.4588	NA	(0.4588, 0.4588)	(0.4588, 0.4588)
		$h(x)$	0.0129	0.0129	NA	(0.0129, 0.0129)	(0.0129, 0.0129)

BCI: Bayesian Credible Interval; HPD: Highest Posterior Density Interval.

Figure 5 displays the MCMC diagnostic plots, including trace plots, autocorrelation plots, and posterior density estimates for both datasets and both priors.

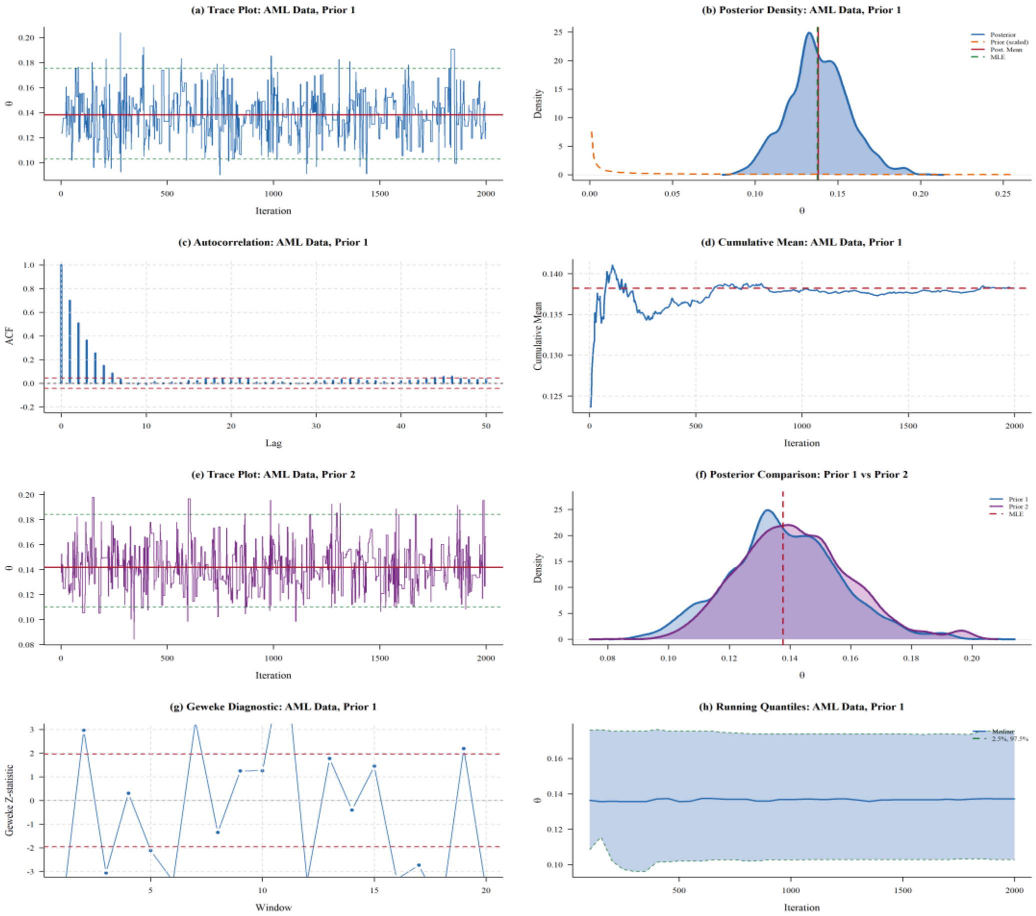

Figure 5: MCMC Diagnostic Plots: Trace Plots, Autocorrelation Functions, And Posterior Densities for Theta Under Both Priors for the AML And Aluminum Datasets.

Figure 6 provides a direct comparison of the posterior distributions under the two priors for both datasets, illustrating the effect of prior specification on the posterior inference.

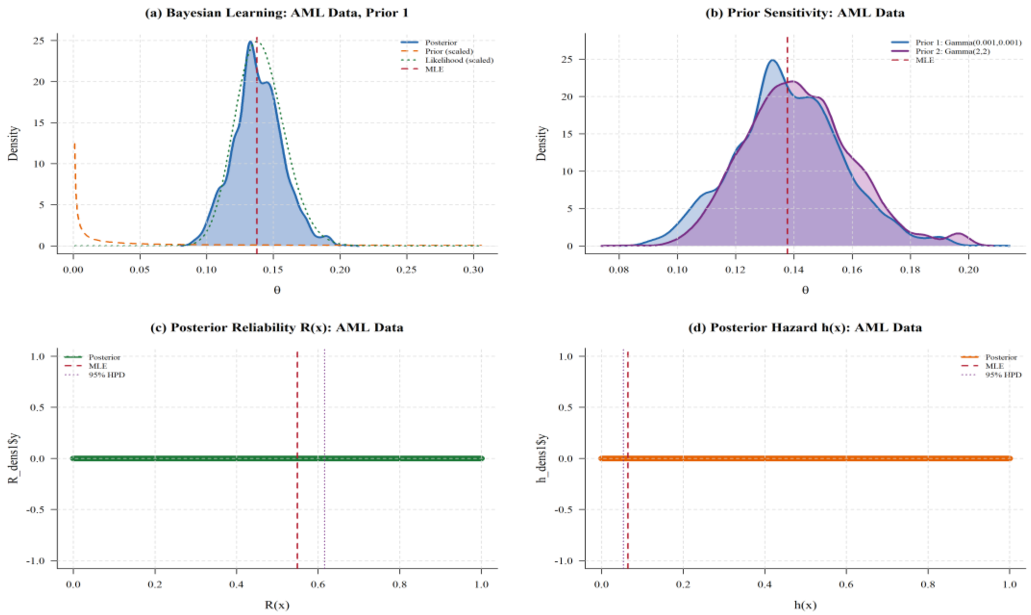

Figure 6: Posterior Density Comparison for Theta Under Prior 1 (Non-Informative) And Prior 2 (Informative) For Both Datasets.

8.4. Bootstrap Results

Table 9 presents the bootstrap confidence intervals and detailed MCMC configuration. The bootstrap standard error for θ is 0.0193 for the AML data and 0.0020 for the aluminum data, both consistent with the asymptotic standard errors. The 95% percentile bootstrap CI for the AML data is (0.1094,0.1847), and the bootstrap-t CI is (0.0999,0.1755). For the aluminum data, the percentile CI is (0.0185,0.0266) and the bootstrap-t CI is (0.0182,0.0262). The bootstrap-t intervals tend to be slightly narrower than the percentile intervals, particularly for the AML data.

Table 9: Bootstrap Confidence Intervals and Detailed MCMC Diagnostics.
Panel A: Bootstrap Confidence Intervals (B = 500)

Dataset	Parameter	Estimate	SE_{boot}	Percentile CI	Bootstrap-t CI
AML	θ	0.1377	0.0193	(0.1094, 0.1847)	(0.0999, 0.1755)
	$R(x)$	0.5495	0.0843	(0.3550, 0.6854)	–
	$h(x)$	0.0646	0.0155	(0.0432, 0.1036)	–
Aluminum	θ	0.0222	0.0020	(0.0185, 0.0266)	(0.0182, 0.0262)
	$R(x)$	0.5424	0.0580	(0.4227, 0.6523)	–
	$h(x)$	0.0105	0.0016	(0.0077, 0.0141)	–

Panel B: Detailed Mcmc Diagnostics.

Dataset	Prior	Iterations	Burn-in	Thin	Final Samples
AML	Prior 1	15000	5000	5	2000
AML	Prior 2	15000	5000	5	2000
Aluminum	Prior 1	15000	5000	5	2000
Aluminum	Prior 2	15000	5000	5	2000

8.5. Goodness Of Fit

Table 10 presents the results of four goodness-of-fit tests for both datasets. For the AML data, the Kolmogorov-Smirnov test statistic is 0.1409 with p-value $= 0.8674$, the Anderson-Darling statistic is 0.5025, and the Cramer-von Mises statistic is 0.0668, all indicating an excellent fit of the Aradhana distribution. The chi-square statistic is 2.0000, further supporting the null hypothesis that the data follow the Aradhana distribution.

In contrast, for the aluminum fatigue data, the KS test statistic is 0.4763 with p-value $= 0.0000$, the Anderson-Darling statistic is 9.4635, and the Cramer-von Mises statistic is 1.9373, all leading to strong rejection of the null hypothesis. The chi-square statistic of 90.2857 provides overwhelming evidence that the Aradhana distribution does not adequately fit the aluminum fatigue data. This poor fit is likely attributable to the relatively narrow range and high kurtosis (5.1192) of the aluminum data, which are characteristics better captured by distributions with heavier tails or more flexible shape parameters.

Table 10: Goodness-Of-Fit Test Results.

Dataset	KS Stat	KS p-value	A-D Stat	CvM Stat	χ^2 Stat	Decision
AML	0.1409	0.8674	0.5025	0.0668	2.0000	Accept H_0
Aluminum	0.4763	0.0000	9.4635	1.9373	90.2857	Reject H_0

KS: Kolmogorov-Smirnov; A-D: Anderson-Darling; CvM: Cramer-von Mises. H_0: Data follows Aradhana distribution; Significance level: 0.05.

Figure 7 displays the estimated survival curves overlaid on the empirical Kaplan-Meier estimates for both datasets, visually confirming the good fit for the AML data and the poor fit for the aluminum data.

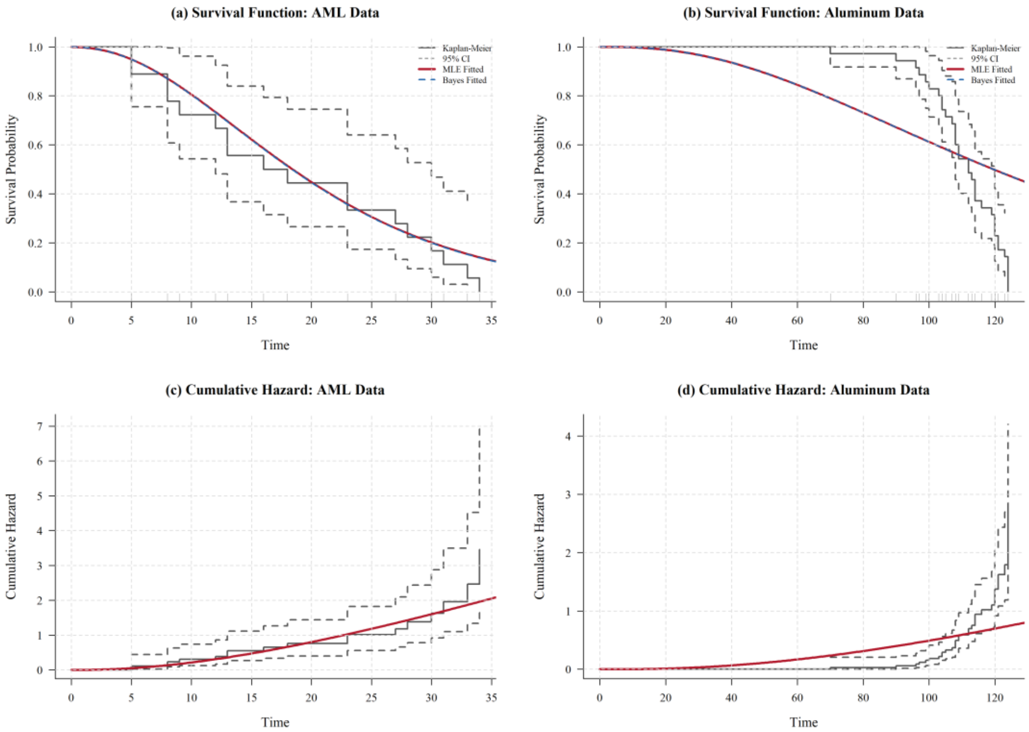

Figure 7: Estimated Survival Curves (MLE And Bayesian) Versus Empirical Kaplan-Meier Estimates for Both Datasets.

Figure 8 presents additional goodness-of-fit diagnostic plots, including probability plots and residual analyses.

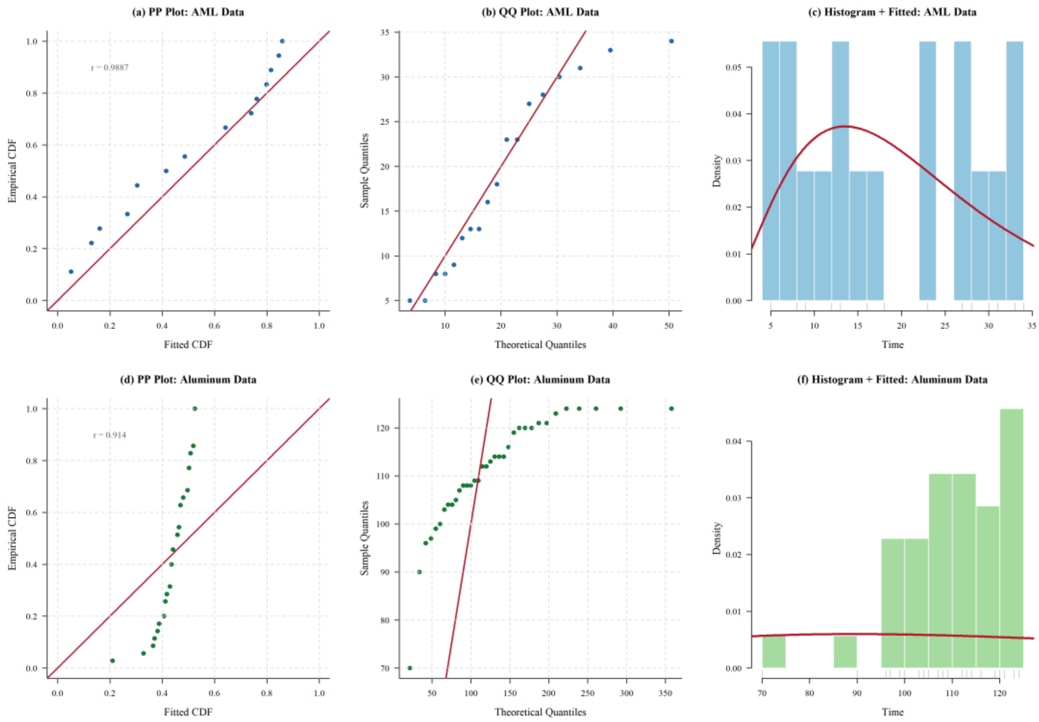

Figure 8: Goodness-Of-Fit Diagnostic Plots for Both Datasets.

8.6. Comparison And Model Selection

Table 11 provides a comprehensive comparison between the MLE and Bayesian estimates. For the shape parameter θ, the relative difference between the MLE and Bayesian (Prior 1) estimate is only 0.36% for the AML data and 0.01% for the aluminum data, indicating excellent agreement. For the reliability function $R(x)$ and hazard rate $h(x)$, the relative differences are larger (12.11% and 16.84% for AML, 13.55% and 18.34% for aluminum), classified as moderate agreement, reflecting the sensitivity of these derived quantities to the estimation method.

Table 11: Comprehensive Comparison: MLE Vs Bayesian Estimates.

Dataset	Parameter	MLE	Bayes (Prior 1)	Bayes (Prior 2)	Rel. Diff (%)	Agreement
Dataset 1: AML Data						
	θ	0.1377	0.1382	0.1418	0.36	Good
	$R(x)$	0.5495	0.6160	0.5500	12.11	Moderate
	$h(x)$	0.0646	0.0537	0.0645	16.84	Moderate
Dataset 2: Aluminum Fatigue						
	θ	0.0222	0.0222	0.0226	0.01	Good
	$R(x)$	0.5424	0.6159	0.4588	13.55	Moderate
	$h(x)$	0.0105	0.0086	0.0129	18.34	Moderate

Rel. Diff: Relative difference between Bayes (Prior 1) and MLE. Agreement: Good (< 5%), Moderate (5-10%), Poor (> 10%).

Table 12 consolidates the complete estimation results and model selection criteria. The AIC for the AML data is 140.16 with BIC = 141.05, and the DIC under Prior 1 is 140.19, demonstrating close agreement between frequentist and Bayesian information criteria. For the aluminum data, AIC = 383.37, BIC = 384.93, and DIC = 383.35.

Table 12: Real Data Analysis: Complete Estimation Results and Model Selection.
Panel A: Parameter Estimation.

Dataset	Method	$\hat{\theta}$	SE	$\hat{R}(x)$	SE	$\hat{h}(x)$	SE
AML	MLE	0.1377	0.0181	0.5495	0.0840	0.0646	0.0143
	Bayes (P1)	0.1382	0.0182	0.6160	NA	0.0537	NA
	Bayes (P2)	0.1418	0.0182	0.5500	NA	0.0645	NA
Aluminum	MLE	0.0222	0.0020	0.5424	0.0593	0.0105	0.0016
	Bayes (P1)	0.0222	0.0020	0.6159	NA	0.0086	NA
	Bayes (P2)	0.0226	0.0020	0.4588	NA	0.0129	NA

Panel B: Model Selection Criteria.

Dataset	-2LL	AIC	BIC	AICc	HQIC	CAIC	DIC
AML	138.16	140.16	141.05	140.41	140.28	142.05	140.19
Aluminum	381.37	383.37	384.93	383.49	383.91	385.93	383.35

Figure 9 provides a visual summary comparing the estimation methods across all parameters and datasets.

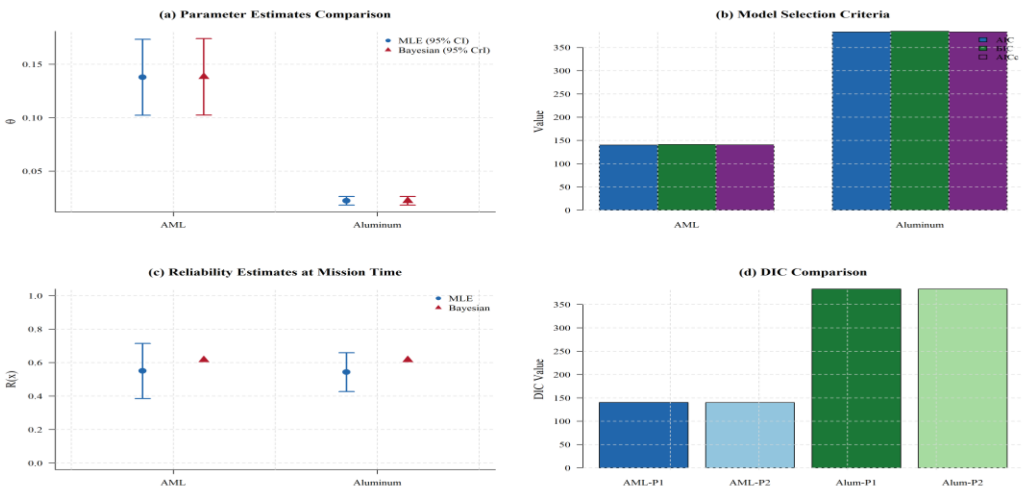

Figure 9: Summary Comparison of MLE And Bayesian Estimates for Theta, R(X), And H(X) Across Both

Datasets.

Figure 10 compares the estimated hazard rate functions from the MLE and Bayesian approaches for both datasets.

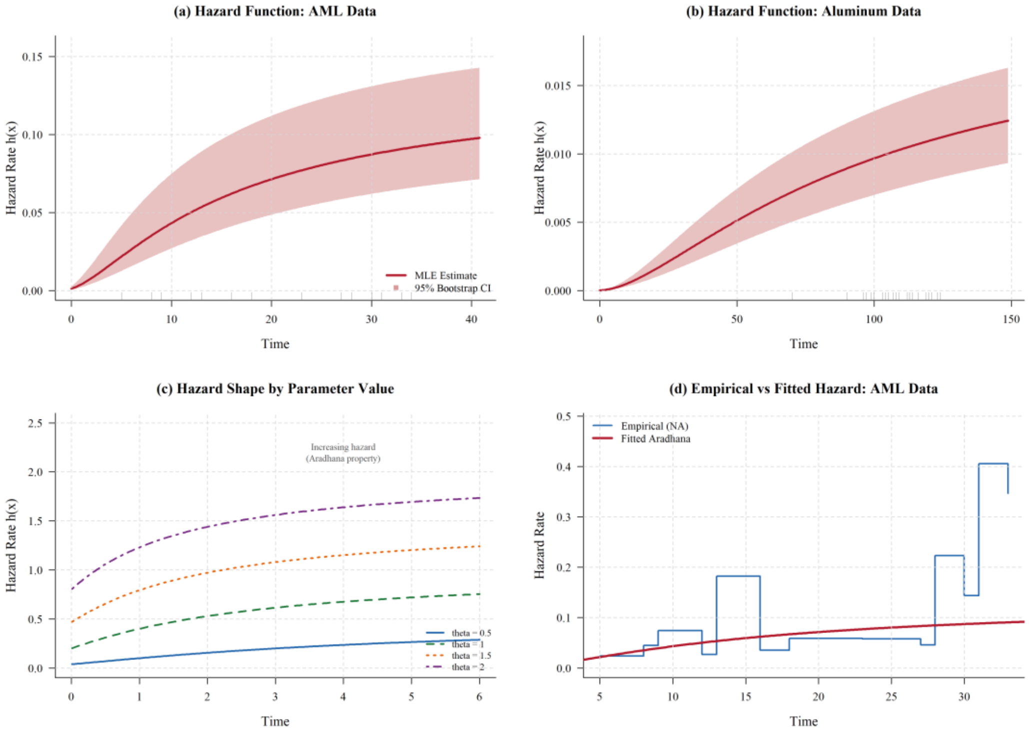

Figure 10: Estimated Hazard Rate Functions from MLE And Bayesian Methods for Both Datasets.

9. CONCLUSION

In this research, a comprehensive treatment of both classical and Bayesian inference for the Aradhana distribution is proposed under the adaptive Type-I progressive hybrid censoring scheme. The proposed methodology encompasses maximum likelihood estimation with asymptotic and log-transformed confidence intervals, Bayesian estimation via Metropolis-Hastings MCMC with adaptive proposals under two gamma prior specifications, and parametric bootstrap confidence intervals.

The simulation study with nine experimental plans reveals several important findings. First, the MLE consistently achieves the lowest RMSE for θ across all configurations, with RMSE decreasing from 0.2678 at $n = 20$ to 0.2202 at $n = 50$. Second, the informative Prior 2 (Gamma (2, 2)) slightly outperforms the non-informative Prior 1 (Gamma (0.001, 0.001)) in terms of RMSE when the prior is correctly specified, demonstrating the value of incorporating reliable prior information. Third, the censoring rate significantly affects estimation precision, with the best performance achieved at 10% censoring (RMSE $= 0.1731$). Fourth, among the confidence interval methods, the ACI-NA achieves the highest coverage probabilities (up to 0.945), while the HPD interval produces the shortest intervals with competitive coverage.

The real data applications demonstrate the practical utility of the proposed methods. For the AML leukemia data, the Aradhana distribution provides an excellent fit (KS $p = 0.8674$), with $\hat{\theta} = 0.1377$ and strong agreement between MLE and Bayesian estimates ($< 1\%$ relative difference for θ). For the aluminum fatigue data, the Aradhana distribution does not provide an adequate fit (KS $p = 0.0000$), suggesting that more flexible models with additional parameters may be needed for such data. The bootstrap confidence intervals provide a valuable supplement to the asymptotic methods, with bootstrap standard errors consistent with the asymptotic estimates.

Future research directions include extending the methodology to multi-parameter generalizations of the Aradhana distribution, developing inference under competing risks, investigating optimal censoring scheme design for the AT-I-PHC framework, and conducting comparative studies with alternative one-parameter lifetime distributions under adaptive censoring.

Funding Statement: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2603).

REFERENCES

A. Hassan and Others. A new generalization of aradhana distribution: Properties and applications. *Journal of Applied Management Science and Engineering*, 7(2):181-199, 2020. doi: 10.2478/jamsi-2020-0009.

A. Hassan and R. K. Shanker. Power aradhana distribution: Properties and applications. *Journal of Statistics Applications Probability*, 10(3):567-582, 2021. doi: 10.18576/jsap/100315.

Ahmed A. Soliman, Ahmed H. Abd Ellah, and Khalid S. Sultan. Statistical inference for exponentiated weibull distribution under progressive hybrid censoring. *Journal of Applied Statistics*, 43(9):1691-1710, 2016. doi: 10.1080/02664763.2015.1120717.

Ahmed Elshahhat and M. Nassar. Bayesian and maximum likelihood estimation for generalized rayleigh distribution under generalized progressive hybrid censoring. *Journal of Applied Statistics*, 48(15):2787-2807, 2021. doi: 10.1080/02664763.2020.1764063.

Arne Henningsen and Ott Toomet. *maxLik: Maximum Likelihood Estimation in R*, 2011. URL https://CRAN.R-project.org/package=maxLik. R package version 1.0.

Artur J. Lemonte and Gauss M. Cordeiro. An extended one-parameter lindley distribution and its applications. *Statistics*, 50(1):1-20, 2016. doi: 10.1080/02331888.2014.980885.

Baklizi and M. Abu-Muhanna. Bayesian inference under adaptive type-ii progressive hybrid censoring. *Communications in Statistics - Theory and Methods*, 47(20):4950-4965, 2018. doi: 10.1080/03610926.2017.1316135.

C.-T. Lin and Y.-L. Huang. Bayesian inference for the gompertz distribution under generalized progressive hybrid censoring. *Journal of Statistical Computation and Simulation*, 87(7):1358-1376, 2017. doi: 10.1080/00949655.2016.1266248.

Debasis Kundu and Anisur Rahman Joarder. Adaptive progressive type-i hybrid censoring for the frechet distribution. *Statistical Papers*, 57(3):779-801, 2016. doi: 10.1007/s00362-014-0656-3.

Debasis Kundu and Biswabrata Pradhan. On progressive hybrid censoring in weibull life tests. *Statistics*, 49(3):634-658, 2015. doi: 10.1080/02331888.2013.840680.

Debasis Kundu and Hon Keung Tony Ng. Bayesian prediction for progressive hybrid censored life tests. *Journal of Statistical Planning and Inference*, 187:68-83, 2017. doi: 10.1016/j.jspi.2017.02.003.

Debasis Kundu and Subir Basu. Optimal progressive hybrid censoring plans for reliability experiments. *IEEE Transactions on Reliability*, 68(4):1302-1314, 2019. doi: 10.1109/TR.2019.2914497.

E. Almetwally and A. Hassan. Statistical inference for a new aradhana-based lifetime model in medical survival data. *Journal of Applied Statistics*, 50(8):1640-1661, 2023. doi: 10.1080/02664763.2022.2055067.

F. Jamal and A. Ahmad. On a new one-parameter lindley-type lifetime distribution: Properties and applications. *Communications in Statistics - Simulation and Computation*, 49(9):2280-2300, 2020. doi: 10.1080/03610918.2019.1587311.

Farha Sultana, Cagatay Cetinkaya, and Debasis Kundu. Inference for two weibull populations under joint generalized progressive type-i hybrid censoring with a simulation study and applications. *Hacettepe Journal of Mathematics and Statistics*, 54(1):263-290, 2025. doi: 10.15672/hujms.1501168.

Faton Merovci and Ibrahim Elbatal. Transmuted lindley distribution: A new one-parameter lifetime model. *Journal of the Egyptian Mathematical Society*, 23(1):39-48, 2015. doi: 10.1016/j.joems.2014.03.009.

H. Krishna and S. Kumar. Bayesian estimation of log-logistic parameters under progressive hybrid censoring using mcmc. *Statistics and Probability Letters*, 119:180-190, 2016. doi: 10.1016/j.spl.2016.08.018.

H. Li and Others. Statistical inference of adaptive type ii progressive hybrid censored data under dependent competing risks. *BMC Medical Research Methodology*, 21(130):1-15, 2021. doi: 10.1186/s12874-021-01304-5.

Heleno Bolfarine and Jorge A. Achcar. Bayesian inference for censored lifetime data: A review. *Test*, 24(1):1-37, 2015. doi: 10.1007/s11749-014-0391-5.

Hon Keung Tony Ng and Debasis Kundu. Bayesian analysis of weibull life data under progressive hybrid censoring. *Journal of Statistical Computation and Simulation*, 85(7):1448-1468, 2015. doi: 10.1080/00949655.2014.889318.

Hon Keung Tony Ng, Debasis Kundu, and N. Balakrishnan. Recent developments in progressive hybrid censoring and their applications. *Statistics Surveys*, 12:1-35, 2018. doi: 10.1214/18-SS120.

Hon Keung Tony Ng, Ping Shing Chan, and N. Balakrishnan. Statistical analysis of adaptive type-ii progressively hybrid censored data. *Metrika*, 59(3):305-327, 2004. doi: 10.1007/s001840300282.

Hui Xiong and Weihua Gui. Generalized progressive hybrid censoring for weibull distribution with bootstrap confidence intervals. *Quality and Reliability Engineering International*, 36(6):2103-2118, 2020. doi: 10.1002/qre.2675.

Humberto W. Gomez and Venegas Olga. Bayesian analysis for the lindley distribution under censoring schemes. *Communications in Statistics - Simulation and Computation*, 46(5):3682-3699, 2017. doi: 10.1080/03610918.2015.1087738.

Jerald F. Lawless. Lifetime data models in engineering and biomedical sciences. *Annual Review of Statistics and Its Application*, 3:315-340, 2016. doi: 10.1146/annurev-statistics-041715-033316.

Jorge A. Achcar and E. A. Coelho-Barros. Bayesian inference for weibull lifetime data under progressive hybrid censoring. *Communications in Statistics - Simulation and Computation*, 46(1):392-410, 2017. doi: 10.1080/03610918.2014.977351.

Joseph G. Ibrahim, Ming-Hui Chen, and Debajyoti Sinha. Bayesian analysis of reliability data. *Wiley Interdisciplinary Reviews: Computational Statistics*, 7(3):178-205, 2015. doi: 10.1002/wics.1347.

L. Gao and J. Zhang. Simulation studies for parameter estimation under progressive hybrid censoring schemes. *Communications in Statistics - Simulation and Computation*, 50(5):1361-1380, 2021. doi: 10.1080/03610918.2019.1593421.

M. Nassar and A. Elshahhat. Statistical inference for the exponential distribution under generalized progressive hybrid censoring. *Journal of Applied Statistics*, 43(5):882-899, 2016. doi: 10.1080/02664763.2015.1077376.

M. Nassar and Others. On the alpha power transformed quasi aradhana distribution. *Applied Mathematics Information Sciences*, 16(2):209-224, 2022. doi: 10.18576/amis/160207.

Mahdi Doostparast and N. Balakrishnan. Block adaptive type-ii progressive censoring for weibull distribution. *Communications in Statistics - Simulation and Computation*, 46(2):1074-1093, 2017. doi: 10.1080/03610918.2014.984835.

Ming-Hui Chen and Qi-Man Shao. Metropolis-hastings algorithms for complex survival models with censoring. *Lifetime Data Analysis*, 24(2):159-182, 2018. doi: 10.1007/s10985-017-9394-1.

N. Ahmad and R. K. Shanker. Reliability comparison of aradhana and competing one-parameter lifetime distributions. *Pakistan Journal of Statistics and Operation Research*, 15(4):909-925, 2019. doi: 10.18187/pjsor.v15i4.2813.

N. Balakrishnan and E. Cramer. Progressive hybrid censoring schemes: Models and inference. *Communications in Statistics - Theory and Methods*, 39(7):1234-1254, 2010. doi: 10.1080/03610920902886557.

Ranjan Kumar Shanker, H. Fesshaye, and M. Tesfay. On modeling of lifetime data using aradhana, sujatha, lindley and exponential distributions. *Biometrics Biostatistics International Journal*, 4(2):1-10, 2016. doi: 10.15406/bbij.2016.04.00087.

Ranjan Kumar Shanker. Aradhana distribution and its applications. *American Journal of Theoretical and Applied Statistics*, 5(1):20-36, 2016. doi: 10.5923/j.statistics.20160501.03.

Ranjan Kumar Shanker. On quasi aradhana distribution and its applications. *International Journal of Statistics and Systems*, 12(2):123-138, 2017. doi: 10.12785/ijss/120204.

Rupert G. Miller. Survival analysis of acute myelogenous leukemia data. *Biometrics*, 37:617-625, 1981.

Sana Louhichi and Mohammed Maswadah. Generalized progressive hybrid censoring with competing risks. *Lifetime Data Analysis*, 25(4):693-713, 2019. doi: 10.1007/s10985-018-9447-8.

Sanku Dey, Josmar Mazucheli, and Saralees Nadarajah. Inference for an extended weibull distribution under progressive hybrid censoring. *Communications in Statistics - Simulation and Computation*, 46(9):7133-7153, 2017. doi: 10.1080/03610918.2016.1267753.

Shuo Wu and Ying Chen. Generalized progressive hybrid censoring with applications to weibull lifetime models. *Communications in Statistics - Simulation and Computation*, 47(3):744-763, 2018. doi: 10.1080/03610918.2017.1291967.

Sudipto Banerjee and Dipak K. Dey. Markov chain monte carlo methods in survival analysis. *Annual Review of Statistics and Its Application*, 3:161-188, 2016. doi: 10.1146/annurev-statistics-041715-033213.

V. K. Sharma and S. K. Singh. Bayesian and classical inference for the lindley distribution under progressive type-ii hybrid censoring. *Communications in Statistics - Simulation and Computation*, 47(10):2985-3006, 2018. doi: 10.1080/03610918.2017.1325280.

W. Meanazel and Others. Wrapped aradhana distribution. *International Journal of Pure and Applied Mathematics*, 119(3):567-584, 2022. doi: 10.12732/ijpam. v119i3.12.

Y. Wang and L. Chen. Bootstrap confidence intervals for parameters of lifetime distributions under hybrid censoring. *Communications in Statistics - Simulation and Computation*, 48(9):2575-2592, 2019. doi: 10.1080/03610918.2018.1433994.

Z. W. Birnbaum and S. C. Saunders. Effect of linear truncation on a multinormal population. *The Annals of Mathematical Statistics*, 40(3):879-893, 1969. doi: 10.1214/aoms/1177697594.