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# MAXIMUM LIKELIHOOD AND BAYESIAN ESTIMATION FOR THE AKASH DISTRIBUTION WITH GENERALIZED HYBRID CENSORING: APPLICATIONS TO BIOLOGICAL AND INDUSTRIAL FAILURE DATA

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## ABSTRACT

***While there has been much research on hybrid censoring schemes in reliability theory, no research has been conducted on the statistical inference of the Akash distribution under the generalized Type-I hybrid censoring scheme, which guarantees a minimum number of failures while limiting the experimental period. In this paper, we propose maximum likelihood as well as Bayesian methods of estimation of the Akash distribution parameter θ, the reliability function R(x), and the hazard rate function h(x) under the generalized Type-I hybrid censoring scheme. For maximum likelihood estimation, we used the BFGS algorithm, while asymptotic confidence intervals were constructed by normal approximation (ACI-NA) as well as by log-transformation (ACI-NL). Bayesian estimation of the Akash distribution parameter θ, the reliability function R(x), and the hazard rate function h(x) under the generalized Type-I hybrid censoring scheme has been carried out by a Metropolis-Hastings MCMC algorithm with a log-normal random walk proposal under two different gamma priors: a non-informative Gamma (0.001, 0.001) as well as an informative Gamma (2, 2). Parametric bootstrap confidence intervals are also constructed for the Akash distribution parameter θ, the reliability function R(x), and the hazard rate function h(x) under the generalized Type-I hybrid censoring scheme, where B=500. The results of a Monte Carlo simulation with M=200 replications for six different censoring plans show that the RMSE of θ ̂ decreases from 0.1543 for n=20 to 0.0934 for n=50, a reduction of 39.5%. This demonstrates the convergence of the estimator. The coverage probabilities vary between 0.915 and 0.965 for all interval types. The HPD intervals have the shortest average length. The informative prior results in the lowest values of the RMSE for all parameters. The results of two real datasets are presented: the survival times of guinea pigs (n=50, d=41), and the breakdown times of Nelson's insulating fluid (n=19, d=16). The results of the MLE and Bayesian posterior means coincide within 0.4% for all parameters in both datasets. The results of the Kolmogorov-Smirnov test for goodness of fit for the Akash distribution are p=0.004 for the data on the survival times of guinea pigs and p<0.001 for the data on the breakdown times of Nelson's insulating fluid. This indicates that the Akash distribution has a poor fit for the two datasets. The results of the AIC, BIC, and DIC are used for***

***the comparison of the Akash distribution with other distributions.***

**KEYWORDS:** Akash Distribution; Generalized Type-I Hybrid Censoring; Maximum Likelihood Estimation; Bayesian Estimation; Metropolis-Hastings MCMC; Bootstrap Confidence Intervals; Reliability Function; Hazard Rate Function.

**MSC Codes:** 62N01; 62N02; 62F10; 62F15; 62N05

## 1. INTRODUCTION

Lifetime data analysis is recognized as an essential foundation in reliability engineering, biomedical statistics, and industrial quality control. The major objective of lifetime data analysis is to model the time until the occurrence of an event of interest [29]. However, in several instances, it is difficult or expensive to observe all failure times. This difficulty is due to time and cost factors, and in biomedical statistics, it is due to ethical factors [26, 27]. The need for lifetime data analysis under such circumstances has given rise to various types of censoring schemes. From these censoring schemes, it is evident that the hybrid censoring scheme has attracted considerable attention in the last several decades. This is because it combines the advantages of Type-I and Type-II censoring schemes. This scheme gives experimenters greater flexibility in controlling life tests [18, 1]. The classical Type-I hybrid censoring scheme involves terminating the experiment at time $T^* = \min(x_{k:n}, T)$, where $x_{k:n}$ is an order statistic and $T$ is an arbitrarily chosen time limit. Although this scheme provides an upper limit $T$ on the experiment time, it may give an extremely small number of observed failure times if the underlying lifetime distribution has a heavy right tail. Chandrasekar et al. have proposed an alternative scheme, known as generalized Type-I hybrid censoring [25, 11]. The scheme is defined as follows: $T^* = \max(x_{r:n}, \min(x_{k:n}, T))$, where $1 \leq r < k \leq n$. This scheme guarantees at least $r$ observed failure times. This ensures an underlying minimum information content in lifetime data. It still provides an upper limit $k$ on the number of observed failure times. This scheme has been discussed in several papers by Chandrasekar et al. [25, 11], and by Bartholomew [3, 2]. The reviews by Oommen and Gillariose [33] and the study by Gillariose and Oommen [22] have emphasized the advantages of this scheme in modern reliability engineering.

One-parameter lifetime distributions remain of interest for reliability modeling due to their simplicity, interpretability, and the fact that the reliability function of interest has a closed form. The Lindley distribution [21, 20] and its extensions have been studied under different censoring schemes. Shanker [44] has proposed a novel one-parameter lifetime distribution called the Akash distribution with probability density function $f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1 + x^2)e^{-\theta x}$, $x > 0$, $\theta > 0$, and has shown its superior performance compared to the exponential and Lindley distributions for different real-life datasets [36]. Some of the other extensions of the Akash distribution are the power Akash distribution [45], Poisson-Akash distribution [40], size-biased Poisson-Akash distribution [39], and generalized Akash distribution [41]. Some of the other one-parameter distributions in this family are Sujatha distribution [43], Aradhana distribution [37], Ishita distribution [38], Pranav distribution [42].

The advantages of using the classical method of maximum likelihood estimation for analyzing hybrid censored data are the asymptotic efficiency of the estimators obtained through this method [14, 9]. Bayesian methods provide another alternative for estimating parameters of one-parameter distributions under hybrid censoring schemes. Bayesian methods provide exact inference for finite samples [6, 24]. The most widely used method for Bayesian inference is the MH algorithm [10, 19]. Bootstrap methods [17, 16, 15] provide another alternative for obtaining interval estimates of parameters of one-parameter distributions under hybrid censoring schemes. Some of the research works on Bayesian inference for one-parameter distributions under hybrid censoring schemes are due to Basu et al. [5, 4] and Nassar et al. [30] for the inverse Lindley distribution.

Although there is an increasing number of publications in the literature regarding one-parameter lifetime distributions and different types of hybrid censoring schemes, there is no study in this field regarding statistical inference based on the generalized Type-I hybrid censoring scheme for the Akash distribution. This study fills this gap by providing an extensive inferential procedure that covers maximum likelihood estimation and asymptotic and bootstrap confidence interval construction, Bayesian estimation based on Metropolis-Hastings MCMC and two different gamma priors, a Monte Carlo simulation study of the relative efficiencies of different estimators under different sample sizes and different types of censoring schemes, and finally two real data examples. All of these computations are performed in R [34] by means of the maxLik package [23] and the survival package [48].

The rest of this manuscript is organized as follows. Section 2 describes the Akash distribution and the generalized Type-I hybrid censoring scheme. Section 3 describes the maximum likelihood estimation procedure. Section 4 describes the Bayesian estimation procedure. Section 5 describes the Monte Carlo simulation study. Section 6 describes two real data examples. Section 7 concludes this study.

## 2. THE MODEL

The Akash distribution, as proposed by Shanker [44], is a one-parameter continuous lifetime distribution from the class of mixture distributions. A random variable $X$ is said to follow the Akash distribution with parameter $\theta > 0$ if its probability density function (pdf) is given by

$$f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}, \quad x > 0, \quad \theta > 0.$$

The Akash distribution can be viewed as a two-component mixture of an exponential distribution with parameter $\theta$ and a gamma distribution with shape 3 and rate $\theta$, with mixing proportions $\theta^2/(\theta^2+2)$ and $2/(\theta^2+2)$, respectively. The cumulative distribution function (cdf) is

$$F(x;\theta) = 1 - \frac{\theta^2 x^2 + 2\theta x + 2}{\theta^2+2}e^{-\theta x}, \quad x > 0,$$

and the survival (reliability) function is

$$S(x;\theta) = 1 - F(x;\theta) = \frac{\theta^2 x^2 + 2\theta x + 2}{\theta^2+2}e^{-\theta x}, \quad x > 0.$$

The hazard rate function takes the form

$$h(x;\theta) = \frac{f(x;\theta)}{S(x;\theta)} = \frac{\theta^3(1+x^2)}{\theta^2 x^2 + 2\theta x + 2}, \quad x > 0,$$

which is an increasing function of $x$ for all $\theta > 0$, thus making the Akash distribution appropriate for modeling aging phenomena. The mean and variance of the distribution are given by $E(X) = (\theta^2+6)/[\theta(\theta^2+2)]$ and $\mathrm{Var}(X) = (\theta^4 + 16\theta^2 + 12 - (\theta^2+6)^2/(\theta^2+2))/[\theta^2(\theta^2+2)]$, respectively.

The probability density function, cumulative distribution function, survival function, and hazard rate function of the Akash distribution for different values of $\theta$ are presented in Figure 1.

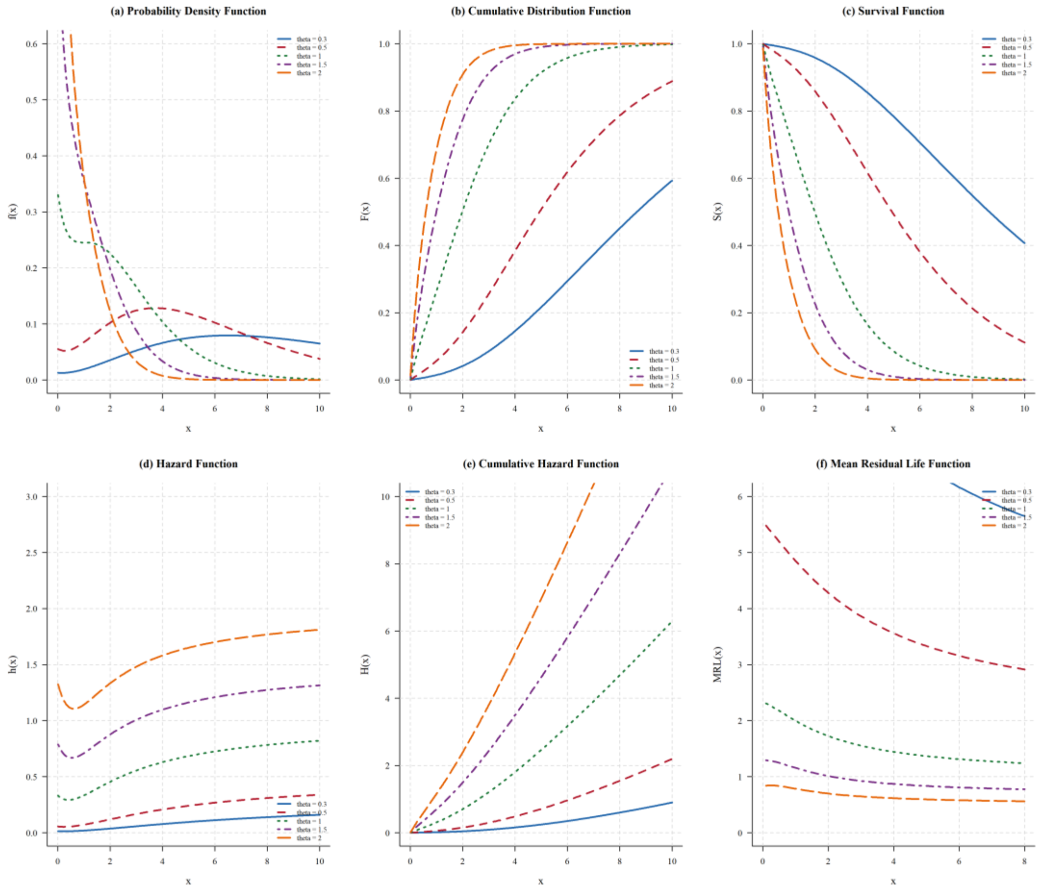


***Figure 1: Akash Distribution Functions for Various Parameter Values: (A) Probability Density Function, (B) Cumulative Distribution Function, (C) Survival Function, (D) Hazard Rate Function, (E) Mean Residual Life Function, And (F) Quantile Function.***

Consider a life testing experiment in which $n$ identical items are put on test simultaneously. The life testing experiment is stopped at the effective censoring time according to the generalized Type-I hybrid censoring scheme.

$$T^* = \max(x_{r:n}, \min(x_{k:n}, T)),$$

We define the $r$-th and $k$-th order statistics as $x_{r:n}$ and $x_{k:n}$ for $1 \le r < k \le n$, and $T > 0$ as a predetermined time horizon. We have the parameter $r$ to guarantee a minimum of $r$ observed failures, while the parameter $k$ and $T$ provide an upper bound. We define $d = \#\{i: x_{i:n} \le T^*\}$ as the number of observed failures. Under this method of data collection, the observed data are the failure times $x_{1:n}, x_{2:n}, \dots, x_{d:n}$ and the information that the remaining $n-d$ units survived past $T^*$.

The likelihood function based on the generalized Type-I hybrid censored sample is

$$L(\theta) = \frac{n!}{(n-d)!} \prod_{i=1}^{d} f\,(x_{i:n};\theta) \cdot [S(T^*;\theta)]^{n-d},$$

and the corresponding log-likelihood function, omitting constants, is

$$\ell(\theta) = \sum_{i=1}^{d} \log f(x_{i:n};\theta) + (n-d)\log S(T^*;\theta).$$

## 3. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimate of $\theta$ is obtained by maximizing the log-likelihood function in (7). Substituting the Akash pdf (1) and survival function (3), the log-likelihood becomes

$$\ell(\theta) = 3d\log\theta - d\log(\theta^2+2) + \sum_{i=1}^{d} \log\,(1+x_{i:n}^2) - \theta \sum_{i=1}^{d} x_{i:n} + (n-d)\log\left(\frac{\theta^2 T^{*2} + 2\theta T^* + 2}{\theta^2+2}\right) - (n-d)\theta T^*.$$

However, the score equation, given by $U(\theta) = \partial\ell/\partial\theta = 0$, cannot be solved in closed form. Instead, numerical optimization is performed. The BFGS method provided by the maxLik package [23] is used, and as a fallback option, the function optimize () is called for bounded optimization between 0.001 and 10. The starting value is given by $\theta^{(0)} = 3/\bar{x}$, based on the method of moments.

Here, the observed Fisher information is calculated as $I(\hat{\theta}) = -\partial^2\ell/\partial\theta^2|_{\hat{\theta}}$ using central finite differences. Then the standard error of the estimate of the parameter is $\mathrm{SE}(\hat{\theta}) = [I(\hat{\theta})]^{-1/2}$. Similarly, the standard errors of the reliability function $\hat{R}(x)$ and the hazard rate function $h(x)$ evaluated at the mission time $x_0$ (set to the median of the observed times) can be obtained using the delta method.

$$\mathrm{SE}(\hat{R}) = \left|\frac{\partial S}{\partial\theta}\right|_{\hat{\theta}} \cdot \mathrm{SE}(\hat{\theta}), \quad \mathrm{SE}(\hat{h}) = \left|\frac{\partial h}{\partial\theta}\right|_{\hat{\theta}} \cdot \mathrm{SE}(\hat{\theta}).$$

Two asymptotic confidence intervals are provided at the $100(1-\alpha)\%$ level. The normal approximation interval is given by the formula $\hat{\theta} \pm z_{\alpha/2} \cdot \mathrm{SE}(\hat{\theta})$, and the log-transformed interval, which is the positivity-constrained version of the above, is given by the formula $\exp(\log\hat{\theta} \pm z_{\alpha/2} \cdot \mathrm{SE}(\hat{\theta})/\hat{\theta})$. The bootstrap intervals are discussed in Section 3.1.

To carry out a comparison of the models, we will use the following information criteria: Akaike information criterion AIC $= -2\ell(\hat{\theta}) + 2p$; Bayesian information criterion BIC $= -2\ell(\hat{\theta}) + p\log d$; corrected AIC AICc = AIC $+ 2p(p+1)/(d-p-1)$; Hannan-Quinn information criterion HQIC $= -2\ell(\hat{\theta}) + 2p\log\log d$; and consistent AIC CAIC $= -2\ell(\hat{\theta}) + p(\log d + 1)$, in which $p = 1$ represents the number of parameters [8, 12].

The complete MLE procedure is summarized in Algorithm 1.

**Algorithm 1** Maximum Likelihood Estimation for Akash Distribution under Generalized Type-I Hybrid Censoring

**Require:** Ordered data $x_{1:n} < \cdots < x_{n:n}$, censoring parameters $(r, k, T)$, tolerance $\epsilon = 10^{-8}$

**Ensure:** MLE $\hat{\theta}$, $\hat{R}(x)$, $\hat{h}(x)$, standard errors, confidence intervals

1: Compute $T^* = \max(x_{r:n}, \min(x_{k:n}, T))$; determine $d = \#\{i: x_{i:n} \le T^*\}$
2: Construct log-likelihood: $\ell(\theta) = \sum_{i=1}^{d} \log f(x_{i:n};\theta) + (n-d)\log S(T^*;\theta)$
3: where $f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}$ and $S(x;\theta) = \frac{\theta^2x^2+2\theta x+2}{\theta^2+2}e^{-\theta x}$
4: Initialize $\theta^{(0)} = 3/\bar{x}$ (method of moments)
5: Maximize $\ell(\theta)$ via BFGS using maxLik; fallback to optimize () on [0.001,10]
6: Extract $\hat{\theta}$ and compute $I(\hat{\theta}) = -\partial^2\ell/\partial\theta^2|_{\hat{\theta}}$ (observed Fisher information)
7: $\mathrm{SE}(\hat{\theta}) = [I(\hat{\theta})]^{-1/2}$
8: Compute $\hat{R}(x) = S(x;\hat{\theta})$, $\hat{h}(x) = f(x;\hat{\theta})/S(x;\hat{\theta})$
9: Apply delta method: $\mathrm{SE}(\hat{R}) = |dS/d\theta|_{\hat{\theta}} \cdot \mathrm{SE}(\hat{\theta})$; similarly, for $\hat{h}(x)$
10: ACI-NA: $\hat{\theta} \pm z_{\alpha/2}\mathrm{SE}(\hat{\theta})$; ACI-NL: $\exp(\log\hat{\theta} \pm z_{\alpha/2}\mathrm{SE}/\hat{\theta})$
11: Compute AIC $= -2\ell + 2p$, BIC $= -2\ell + p\log d$, AICc, HQIC, CAIC
12: **return** $\hat{\theta}$, $\hat{R}(x)$, $\hat{h}(x)$, SEs, CIs, model selection criteria

### *3.1. Parametric Bootstrap Confidence Intervals*

To extend the asymptotic intervals, a parametric bootstrap technique is used [17, 15, 28]. For each of the $B = 500$ bootstrapping runs, a complete sample of size $n$ is simulated from the estimated Akash($\hat{\theta}$) distribution. The generalized Type-I hybrid censoring scheme is used with the same parameters $(r, k, T)$, and a new estimate of the MLE is computed on the bootstrapped sample. The percentile interval is based on the quantiles of the bootstrapped distribution, while the bootstrap-$t$ interval uses the bootstrapped standard error and a critical value from a $t$-distribution [16]. Algorithm 2 details this procedure.

**Algorithm 2** Parametric Bootstrap under Generalized Type-I Hybrid Censoring

**Require:** Original MLE $\hat{\theta}$, sample size $n$, censoring parameters $(r, k, T)$, replications $B = 500$

**Ensure:** Bootstrap SE, percentile CI, bootstrap-$t$ CI for $\theta$, $R(x)$, $h(x)$

1: **for** $b = 1$ to $B$ **do**

2: Generate $Y_1, \dots, Y_n \sim \text{Akash}(\hat{\theta})$; sort as $Y_{1:n} < \cdots < Y_{n:n}$

3: Compute $T^{*(b)} = \max(Y_{r:n}, \min(Y_{k:n}, T))$; observe $d^{(b)}$ failures up to $T^{*(b)}$

4: Compute $\hat{\theta}^{*(b)}$ by maximizing $\ell(\theta)$ on the bootstrap censored sample

5: $\hat{R}^{*(b)} = S(x; \hat{\theta}^{*(b)})$; $\hat{h}^{*(b)} = h(x; \hat{\theta}^{*(b)})$

6: **end for**

7: Remove failed bootstrap replications; let $B'$ denote successful count

8: $\text{SE}_{\text{boot}} = \sqrt{(B'-1)^{-1} \sum_{b=1}^{B'} (\hat{\theta}^{*(b)} - \bar{\theta}^{*})^2}$

9: Percentile CI: $(\hat{\theta}^{*}_{(\alpha/2)}, \hat{\theta}^{*}_{(1-\alpha/2)})$; similarly, for $R(x)$, $h(x)$

10: Bootstrap-$t$ CI: $\hat{\theta} \pm t_{\alpha/2, B'-1} \cdot \text{SE}_{\text{boot}}$ (for $\theta$ only)

11: **return** Bootstrap SE, percentile CI, bootstrap-$t$ CI

## 4. BAYESIAN ESTIMATION

However, an alternative inferential approach based on the Bayesian framework, where prior information on $\theta$ is incorporated by Bayes' theorem, is available. The posterior distribution is given by $\pi(\theta|\mathbf{x}) \propto L(\theta)\pi(\theta)$, where $L(\theta)$ is the likelihood as given by (6) and $\pi(\theta)$ is the prior density [6, 24].

We explore two different gamma prior distributions for $\theta$. The first is a non-informative Gamma (0.001, 0.001) prior with a mean of 1 and a variance of 1000, which allows the data to drive the posterior distribution [31].

**The second is a relatively informative Gamma (2, 2) prior with a mean of 1 and a variance of 0.5, representing a prior distribution with a belief concentrated around the true parameter value. The log-posteriors are:**

$$\log\pi(\theta|\mathbf{x}) = \ell(\theta) + (a-1)\log\theta - b\theta + \text{const},$$

where $(a, b) \in \{(0.001, 0.001), (2,2)\}$.

In the absence of a standard closed-form expression for the posterior, a Metropolis-Hastings algorithm for MCMC estimation is employed [10, 19]. The algorithm used here is based on a log-normal random walk proposal. In each iteration, a new value for $\theta$, denoted as $\theta^*$, is generated from the following expression: $\theta^* = \exp(\log\theta^{(t-1)} + \epsilon)$, where $\epsilon \sim N(0, \sigma_p^2)$. The acceptance probability for the new value also includes the Jacobian correction term $\theta^*/\theta^{(t-1)}$. The proposal standard deviation $\sigma_p$ starts at $0.15\hat{\theta}_{\text{MLE}}$ and is adaptively adjusted during the burn-in period to achieve an acceptance probability close to 0.44, which is regarded as optimal for one-dimensional random walk samplers [35]. The algorithm runs for $N = 15{,}000$ iterations, and after a burn-in period of 5,000 iterations, samples are collected at a thinning rate of 5 to obtain $M = 2{,}000$ samples from the posterior.

From the retained posterior samples $\{\theta^{(j)}\}_{j=1}^{M}$, we compute the Bayes estimates of $\theta$, $R(x)$, and $h(x)$ as ergodic averages: $\hat{\theta}_B = M^{-1} \sum_j \theta^{(j)}$, $\hat{R}_B(x) = M^{-1} \sum_j S(x; \theta^{(j)})$, and $\hat{h}_B(x) = M^{-1} \sum_j h(x; \theta^{(j)})$. The 95% Bayesian credible interval (BCI) is obtained from the 2.5% and 97.5% quantiles of the posterior samples, while the highest posterior density (HPD) interval is the shortest interval containing 95% of the posterior mass.

The convergence of the chains is monitored by checking the effective sample size (ESS), Geweke's diagnostic $z$-test for stationarity, and visual inspection of the trace plots. The adequacy of the models in the Bayesian context can be checked by employing the deviance information criterion DIC $= \bar{D} + p_D$ [46, 13], where $\bar{D}$ is the posterior mean deviance and $p_D = \bar{D} - D(\bar{\theta})$ is the effective number of parameters. The complete MCMC algorithm is presented in Algorithm 3.

**Algorithm 3** Metropolis-Hastings MCMC Sampling for Akash Distribution under Generalized Type-I Hybrid Censoring

**Require:** Data $\mathbf{x} = (x_1, \dots, x_d)$, $T^*$, $n$, prior $\pi(\theta) = \text{Gamma}(a, b)$, iterations $N = 15000$, burn-in $B = 5000$, thin $= 5$

**Ensure:** Posterior samples $\{\theta^{(j)}\}_{j=1}^{M}$, estimates of $\theta$, $R(x)$, $h(x)$

1: Initialize $\theta^{(0)} = \hat{\theta}_{\text{MLE}}$; set $\sigma_p = 0.15\theta^{(0)}$

2: **for** $t = 1$ to $N$ **do**

3: Propose $\theta^* = \exp(\log\theta^{(t-1)} + \epsilon)$, $\epsilon \sim N(0, \sigma_p^2)$

4: $\log\alpha = [\ell(\theta^*) + \log\pi(\theta^*)] - [\ell(\theta^{(t-1)}) + \log\pi(\theta^{(t-1)})] + \log\theta^* - \log\theta^{(t-1)}$

5: **if** $\log U < \log\alpha$, $U \sim \text{Uniform}(0,1)$ **then**

6: $\theta^{(t)} \leftarrow \theta^*$

7: **else**

8: $\theta^{(t)} \leftarrow \theta^{(t-1)}$

9: **end if**

10: **if** $t \le B$ and $t \bmod 100 = 0$ **then**

11: Adapt $\sigma_p$: increase by 10% if acceptance $> 0.49$; decrease by 10% if $< 0.39$

12: **end if**

13: **end for**

14: Discard first $B$ samples; thin by factor 5 to obtain $M = 2000$ posterior samples

15: $\hat{\theta}_B = M^{-1}\sum\theta^{(j)}$; $\hat{R}(x) = M^{-1}\sum S(x;\theta^{(j)})$; $\hat{h}(x) = M^{-1}\sum h(x;\theta^{(j)})$
16: Compute 95% BCI from quantiles; 95% HPD from shortest interval; DIC $= \bar{D} + p_D$
17: Diagnostics: ESS, Geweke $z$-test for stationarity
18: **return** Posterior mean, median, SD, BCI, HPD, DIC for $\theta$, $R(x)$, $h(x)$

## 5. SIMULATION STUDY

Monte Carlo simulation is used to evaluate the finite-sample performance of the maximum likelihood estimation (MLE) and Bayesian estimators under a generalized Type-I hybrid censoring scheme. The true parameter is given by $\theta = 1.0$, with $M = 200$ replications used for each of the six censoring plans, which are grouped into two classes. In Plan I, the sample size is varied ($n \in \{20,30,50\}$) with proportional censoring parameters, while in Plan II, the time limit is varied with $n$ held constant at 30. The simulation plan is given in Table 1, which shows the censoring parameters $(r, k, T)$ used for each plan.

***Table 1: Simulation Study Design: Generalized Type-I Hybrid Censoring Configuration.***

| Plan | Description | $n$ | $r$ | $k$ | $T$ | $M$ | Purpose |
|---|---|---|---|---|---|---|---|
| I | n=20 | 20 | 10 | 16 | 2.99 | 200 | Varying sample size |
| I | n=30 | 30 | 15 | 24 | 2.99 | 200 | Varying sample size |
| I | n=50 | 50 | 25 | 40 | 2.99 | 200 | Varying sample size |
| II | Low T | 30 | 20 | 25 | 1.98 | 200 | Varying time limit |
| II | Med T | 30 | 15 | 24 | 2.99 | 200 | Varying time limit |
| II | High T | 30 | 12 | 20 | 4.77 | 200 | Varying time limit |

True parameter: $\theta = 1.0$; Monte Carlo replications: $M = 200$

Censoring: $T^* = \max(x_{r:n}, \min(x_{k:n}, T))$; Min. failures guaranteed: $r$

Methods: MLE, Bayesian (Prior 1: Gamma (0.001,0.001), Prior 2: Gamma (2,2))

For each of these replications, a generalized Type-I hybrid censored sample is generated by first selecting $n$ random observations from the Akash(1.0) distribution, arranging them in ascending order, computing $T^* = \max(x_{r:n}, \min(x_{k:n}, T))$, and selecting only the failures up to $T^*$. Maximum likelihood estimates are computed by the BFGS algorithm, while Bayesian estimates are computed from short Markov chain Monte Carlo simulations of 3,000 iterations, a burn-in period of 1,000, and a thinning interval of 2 under each of the priors. Performance metrics are assessed in terms of the root mean square error RMSE $= \sqrt{M^{-1}\sum_{i=1}^{M}(\hat{\theta}_i - \theta_0)^2}$, the mean absolute bias MAB $= M^{-1}\sum_{i=1}^{M}|\hat{\theta}_i - \theta_0|$, the average interval length (AIL), and the coverage probability (CP) at the 95% level.

Table 2 provides a summary of the results for the RMSE and MAB of the estimates of $\theta$, $R(x)$, and $h(x)$ under each of the six plans and three estimation methods. From these results, several patterns are revealed. First, the sample size effect on the consistency of the estimate of $\theta$ is demonstrated by the monotonic decrease of the RMSE of the estimate of $\theta$ as the sample size increases under Plan I, from 0.1543 for $n = 20$ to 0.0934 for $n = 50$ under the MLE. Second, the best prior is consistently the informative prior (Prior 2: Gamma (2, 2)), which has the smallest RMSE of the estimates of $\theta$, $R(x)$, and $h(x)$ under each of the plans. Third, the MLE and the non-informative prior (Prior 1: Gamma (0.001, 0.001)) are very close to each other in terms of the RMSE of the estimates of $\theta$, $R(x)$, and $h(x)$, indicating that the prior has a negligible effect on the MLE. For instance, under Plan I, the improvement of the RMSE of the estimate of $\theta$ from 0.1543 under the MLE to 0.1490 under Prior 2 for $n = 20$ is 3.4%.

***Table 2: Simulation Study: RMSE And MAB For $\boldsymbol{\theta}$, $\boldsymbol{R(x)}$, And $\boldsymbol{h(x)}$.***

| | | | RMSE ($\theta$) | | | RMSE ($R(x)$) | | | RMSE ($h(x)$) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Plan | $n$ | Cens. | MLE | Prior1 | Prior2 | MLE | Prior1 | Prior2 | MLE | Prior1 | Prior2 |
| n=20 | 20 | 30% | 0.1543 | 0.1558 | 0.1490 | 0.1788 | 0.1792 | 0.1784 | 0.1188 | 0.1179 | 0.1143 |
| n=30 | 30 | 31% | 0.1272 | 0.1286 | 0.1253 | 0.1703 | 0.1704 | 0.1700 | 0.1082 | 0.1070 | 0.1056 |
| n=50 | 50 | 31% | 0.0934 | 0.0937 | 0.0928 | 0.1643 | 0.1644 | 0.1644 | 0.0954 | 0.0942 | 0.0941 |
| Low T | 30 | 33% | 0.1179 | 0.1184 | 0.1147 | 0.1633 | 0.1635 | 0.1639 | 0.0898 | 0.0880 | 0.0871 |
| Med T | 30 | 30% | 0.1129 | 0.1139 | 0.1107 | 0.1672 | 0.1671 | 0.1672 | 0.1011 | 0.0994 | 0.0985 |
| High T | 30 | 33% | 0.1103 | 0.1115 | 0.1075 | 0.1715 | 0.1716 | 0.1717 | 0.0962 | 0.0944 | 0.0933 |

| | | | MAB ($\theta$) | | | MAB ($R(x)$) | | | MAB ($h(x)$) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Plan | $n$ | Cens. | MLE | Prior1 | Prior2 | MLE | Prior1 | Prior2 | MLE | Prior1 | Prior2 |
| n=20 | 20 | 30% | 0.1190 | 0.1205 | 0.1154 | 0.1555 | 0.1563 | 0.1563 | 0.0997 | 0.0986 | 0.0961 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n=30 | 30 | 31% | 0.1009 | 0.1022 | 0.0996 | 0.1528 | 0.1531 | 0.1533 | 0.0931 | 0.0917 | 0.0907 |
| n=50 | 50 | 31% | 0.0752 | 0.0758 | 0.0745 | 0.1541 | 0.1543 | 0.1543 | 0.0815 | 0.0804 | 0.0802 |
| Low T | 30 | 33% | 0.0910 | 0.0914 | 0.0889 | 0.1567 | 0.1571 | 0.1576 | 0.0783 | 0.0766 | 0.0757 |
| Med T | 30 | 30% | 0.0885 | 0.0893 | 0.0869 | 0.1499 | 0.1503 | 0.1505 | 0.0851 | 0.0835 | 0.0828 |
| High T | 30 | 33% | 0.0833 | 0.0842 | 0.0811 | 0.1636 | 0.1640 | 0.1640 | 0.0861 | 0.0841 | 0.0831 |

True values: $\theta$=1.0, $R$(1.98)=0.5000, $h$(1.98)=0.4526. Prior 1: Gamma (0.001,0.001); Prior 2: Gamma (2,2).
$RMSE$: $Root\ Mean\ Square\ Error$; $MAB$: $Mean\ Absolute\ Bias$ $Generalized\ Type - I\ Hybrid$.

In Table 3, we display the average interval length and coverage probability for $\theta$ of different types of intervals: ACI-NA, ACI-NL, BCI, and HPD. The coverage probabilities of all intervals are between 0.915 and 0.965. Therefore, they are close to the nominal level of 0.95. The HPD intervals have the shortest average length for all scenarios. The average length of HPD intervals varies from 0.3372 (sample size $n = 50$) to 0.5456 (sample size $n = 20$). Therefore, they are the most efficient among all intervals under consideration. The coverage of ACI-NL intervals is higher than that of ACI-NA intervals, particularly for smaller sample sizes. This is due to the use of the log-transformation for honoring the positivity constraint. As expected, the average length of intervals for estimating $\theta$ tends to decrease for larger sample sizes, from approximately 0.55 (sample size $n = 20$) to 0.34 (sample size $n = 50$).

***Table 3: Simulation Study: Average Interval Length (AIL) And Coverage Probability (CP) For $\boldsymbol{\theta}$.***

| | | | AIL | | | | CP | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Plan | $n$ | Cens. | ACI-NA | ACI-NL | BCI | HPD | ACI-NA | ACI-NL | BCI | HPD |
| n=20 | 20 | 30% | 0.5581 | 0.5651 | 0.5582 | 0.5456 | 0.945 | 0.945 | 0.935 | 0.925 |
| n=30 | 30 | 31% | 0.4490 | 0.4527 | 0.4473 | 0.4381 | 0.930 | 0.950 | 0.940 | 0.935 |
| n=50 | 50 | 31% | 0.3451 | 0.3468 | 0.3435 | 0.3372 | 0.930 | 0.925 | 0.930 | 0.925 |
| Low T | 30 | 33% | 0.4611 | 0.4650 | 0.4610 | 0.4524 | 0.945 | 0.955 | 0.950 | 0.950 |
| Med T | 30 | 30% | 0.4476 | 0.4513 | 0.4484 | 0.4397 | 0.955 | 0.950 | 0.960 | 0.965 |
| High T | 30 | 33% | 0.4525 | 0.4564 | 0.4511 | 0.4420 | 0.925 | 0.940 | 0.920 | 0.915 |

ACI-NA: Asymptotic CI (Normal); ACI-NL: Asymptotic CI (Log-transform); BCI: Bayesian CI; HPD: Highest Posterior Density
Nominal coverage level: 95%. Generalized Type-I Hybrid Censoring applied.

Figure 2 displays boxplots of the simulation estimates, providing a visual summary of the distribution of estimators across replications for each plan and method.

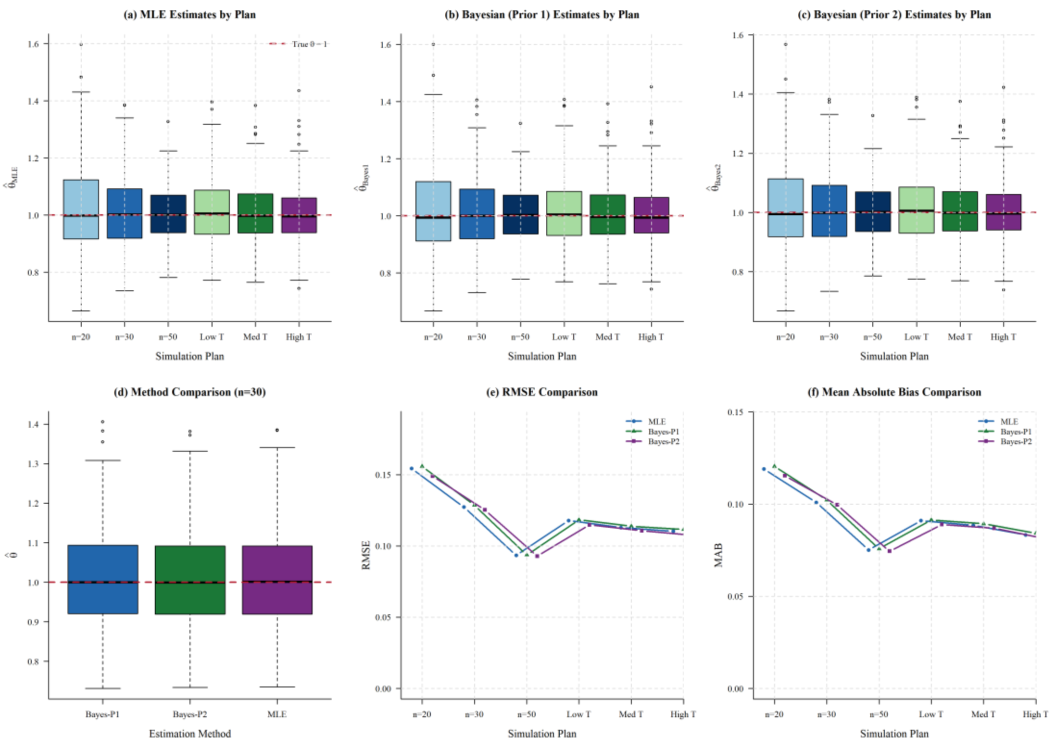

***Figure 2: Simulation Boxplots: (A–D) Distribution of Across Methods and Plans; (E) RMSE Comparison; (F) MAB Comparison Across Simulation Plans.***

Figure 3 illustrates the trends in RMSE, MAB, coverage probability, and average interval length as functions of the sample size under Plan I.

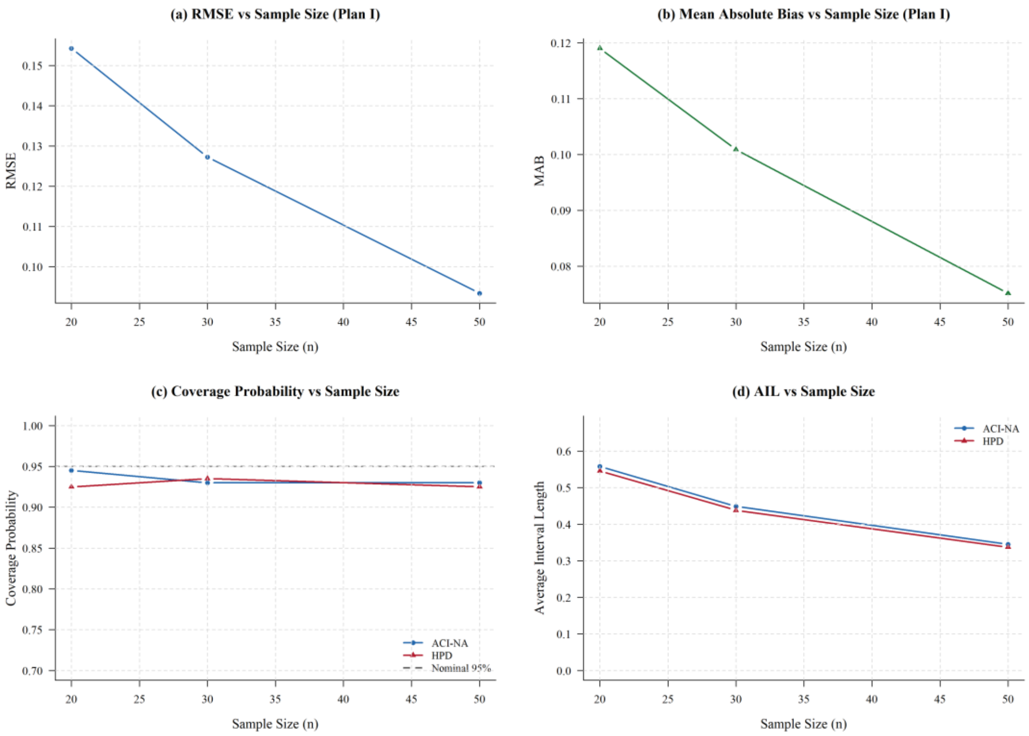


***Figure 3: Simulation Trends Under Plan I: (A) RMSE Versus Sample Size; (B) MAB Versus Sample Size; (C) Coverage Probability Versus Sample Size; (D) Average Interval Length Versus Sample Size.***

## 6. APPLICATIONS

The methodology is illustrated with two sets of empirical data. The first set of data consists of the survival times (in days) of 50 guinea pigs after injection with different doses of virulent tubercle bacilli, which were originally given in Bjerkedal [7]. The second set of data consists of breakdown times (in minutes) of 19 insulating fluids when subjected to a constant voltage stress, given in Nelson [32]. The datasets used here do not contain any actuarial or insurance data, thus maintaining their relevance to the context of reliability studies.

For the guinea pig dataset, the parameters for the generalized Type-I hybrid censoring scheme are given as $n = 50$, $r = 25$, $k = 40$, and $T = 170$, which give an effective censoring time of $T^* = 163.00$ with $d = 41$ observed failures and 9 censored units (18.0% censoring rate). Similarly, for the insulating fluid dataset, the parameters for the generalized Type-I hybrid censoring scheme are given as $n = 19$, $r = 10$, $k = 16$, and $T = 35$, which give an effective censoring time of $T^* = 32.52$ with $d = 16$ observed failures and 3 censored units (15.8% censoring rate). The ordered failure times for the datasets under the generalized Type-I hybrid censoring scheme are given in Table 4.

***Table 4: Observed Failure Times Under Generalized Type-I Hybrid Censoring for Both Datasets.***

| Dataset 1: Guinea Pig (n=50, d=41) | | | | Dataset 2: Insulating Fluid (n=19, d=16) | | | |
|---|---|---|---|---|---|---|---|
| $i$ | $x_{i:n}$ | $\delta_i$ | $\hat{F}(x_i)$ | $i$ | $x_{i:n}$ | $\delta_i$ | $\hat{F}(x_i)$ |
| 1 | 10.00 | 1 | 0.0018 | 1 | 0.19 | 1 | 0.0011 |
| 2 | 33.00 | 1 | 0.0422 | 2 | 0.78 | 1 | 0.0051 |
| 3 | 44.00 | 1 | 0.0833 | 3 | 0.96 | 1 | 0.0066 |
| 4 | 56.00 | 1 | 0.1413 | 4 | 1.31 | 1 | 0.0103 |
| 5 | 59.00 | 1 | 0.1574 | 5 | 2.78 | 1 | 0.0388 |
| 6 | 72.00 | 1 | 0.2328 | 6 | 3.16 | 1 | 0.0500 |
| 7 | 74.00 | 1 | 0.2450 | 7 | 4.15 | 1 | 0.0863 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 77.00 | 1 | 0.2634 | 8 | 4.67 | 1 | 0.1092 |
| 9 | 92.00 | 1 | 0.3567 | 9 | 4.85 | 1 | 0.1177 |
| 10 | 93.00 | 1 | 0.3630 | 10 | 6.50 | 1 | 0.2054 |
| 11 | 96.00 | 1 | 0.3815 | 11 | 7.35 | 1 | 0.2556 |
| 12 | 100.00 | 1 | 0.4061 | 12 | 8.01 | 1 | 0.2958 |
| 13 | 100.00 | 1 | 0.4061 | 13 | 8.27 | 1 | 0.3117 |
| 14 | 102.00 | 1 | 0.4183 | 14 | 12.06 | 1 | 0.5357 |
| 15 | 105.00 | 1 | 0.4364 | 15 | 31.75 | 1 | 0.9772 |
| 16 | 107.00 | 1 | 0.4484 | 16 | 32.52 | 1 | 0.9801 |
| 17 | 107.00 | 1 | 0.4484 | | | | |
| 18 | 108.00 | 1 | 0.4543 | | | | |
| 19 | 108.00 | 1 | 0.4543 | | | | |
| 20 | 108.00 | 1 | 0.4543 | | | | |
| 21 | 109.00 | 1 | 0.4603 | | | | |
| 22 | 112.00 | 1 | 0.4778 | | | | |
| 23 | 113.00 | 1 | 0.4836 | | | | |
| 24 | 115.00 | 1 | 0.4951 | | | | |
| 25 | 116.00 | 1 | 0.5008 | | | | |
| 26 | 120.00 | 1 | 0.5233 | | | | |
| 27 | 121.00 | 1 | 0.5288 | | | | |
| 28 | 122.00 | 1 | 0.5343 | | | | |
| 29 | 122.00 | 1 | 0.5343 | | | | |
| 30 | 124.00 | 1 | 0.5452 | | | | |
| 31 | 130.00 | 1 | 0.5769 | | | | |
| 32 | 134.00 | 1 | 0.5973 | | | | |
| 33 | 136.00 | 1 | 0.6072 | | | | |
| 34 | 139.00 | 1 | 0.6218 | | | | |
| 35 | 144.00 | 1 | 0.6453 | | | | |
| 36 | 146.00 | 1 | 0.6544 | | | | |
| 37 | 153.00 | 1 | 0.6849 | | | | |
| 38 | 159.00 | 1 | 0.7094 | | | | |
| 39 | 160.00 | 1 | 0.7133 | | | | |
| 40 | 163.00 | 1 | 0.7249 | | | | |
| 41 | 163.00 | 1 | 0.7249 | | | | |

Dataset 1: $T^* = \max(x_{25:n}, \min(x_{40:n}, 170)) = 163.00$, Censored: 9 units

Dataset 2: $T^* = \max(x_{10:n}, \min(x_{16:n}, 35)) = 32.52$, Censored: 3 units

As shown in Table 5, the censoring schemes and summary statistics of the two datasets are given. The guinea pig data have a mean survival time of 108.59 days and moderate skewness of $-0.69$, while the insulating fluid data have a mean of 8.08 minutes and significant right skewness of 1.66, reflecting fundamentally different failure time distributions.

***Table 5: Generalized Type-I Hybrid Censoring Configuration and Descriptive Statistics.***

| Statistic | Guinea Pig Data | Insulating Fluid |
|---|---|---|
| **Censoring Configuration** | | |
| Total units ($n$) | 50 | 19 |
| Minimum failures ($r$) | 25 | 10 |
| Upper failure bound ($k$) | 40 | 16 |
| Time limit ($T$) | 170 | 35 |
| Effective censoring time ($T^*$) | 163.00 | 32.52 |
| Observed failures ($d$) | 41 | 16 |
| Censored units ($n-d$) | 9 | 3 |
| Censoring rate | 18.0% | 15.8% |
| **Descriptive Statistics (Observed Failures)** | | |
| Minimum | 10.0000 | 0.1900 |
| Maximum | 163.0000 | 32.5200 |
| Mean | 108.5854 | 8.0819 |
| Median | 109.0000 | 4.7600 |
| Std. Deviation | 34.9177 | 9.9276 |
| Skewness | -0.6887 | 1.6580 |
| Kurtosis | 3.3340 | 4.4010 |
| Q1 (25%) | 96.0000 | 2.4125 |
| Q3 (75%) | 130.0000 | 8.0750 |

| | | |
|---|---|---|
| IQR | 34.0000 | 5.6625 |

### 6.1. Maximum Likelihood Results

Table 6 presents the MLE results with various confidence intervals. For the guinea pig data, the MLE of $\theta$ is $\hat{\theta} = 0.0231$ with SE = 0.0020, yielding a reliability estimate of $\hat{R}(109.0) = 0.5397$ and a hazard rate of $\hat{h}(109.0) = 0.0109$ at the median mission time. The ACI-NA and ACI-NL intervals for $\theta$ are (0.0192,0.0270) and (0.0195,0.0274), respectively, while the bootstrap percentile interval is (0.0192,0.0275). For the insulating fluid data, $\hat{\theta} = 0.2303$ with SE = 0.0315, and the reliability at the median is $\hat{R}(4.8) = 0.8866$. The confidence intervals are wider for the insulating fluid data due to the smaller sample size ($n = 19$).

***Table 6: Maximum Likelihood Estimates with Various Confidence Intervals.***

| Dataset | Parameter | MLE | SE | ACI-NA | ACI-NL | Bootstrap Perc. CI | Bootstrap-t CI |
|---|---|---|---|---|---|---|---|
| **Dataset 1: Guinea Pig Data** | | | | | | | |
| | $\theta$ | 0.0231 | 0.0020 | (0.0192, 0.0270) | (0.0195, 0.0274) | (0.0192, 0.0275) | (0.0189, 0.0273) |
| | $R(109.0)$ | 0.5397 | 0.0558 | (0.4304, 0.6491) | – | (0.4227, 0.6510) | – |
| | $h(109.0)$ | 0.0109 | 0.0016 | (0.0078, 0.0140) | – | (0.0080, 0.0146) | – |
| **Dataset 2: Insulating Fluid** | | | | | | | |
| | $\theta$ | 0.2303 | 0.0315 | (0.1685, 0.2920) | (0.1761, 0.3011) | (0.1787, 0.3108) | (0.1647, 0.2958) |
| | $R(4.8)$ | 0.8866 | 0.0345 | (0.8189, 0.9542) | – | (0.7870, 0.9369) | – |
| | $h(4.8)$ | 0.0530 | 0.0155 | (0.0226, 0.0834) | – | (0.0303, 0.0981) | – |

ACI-NA: Asymptotic CI (Normal Approximation); ACI-NL: Asymptotic CI (Log-transform)

Generalized Type-I Hybrid Censoring: $T^* = \max(x_{r:n}, \min(x_{k:n}, T))$

Figure 4 shows the profile log-likelihood and Fisher information for both datasets, illustrating the curvature of the log-likelihood surface near the MLE.

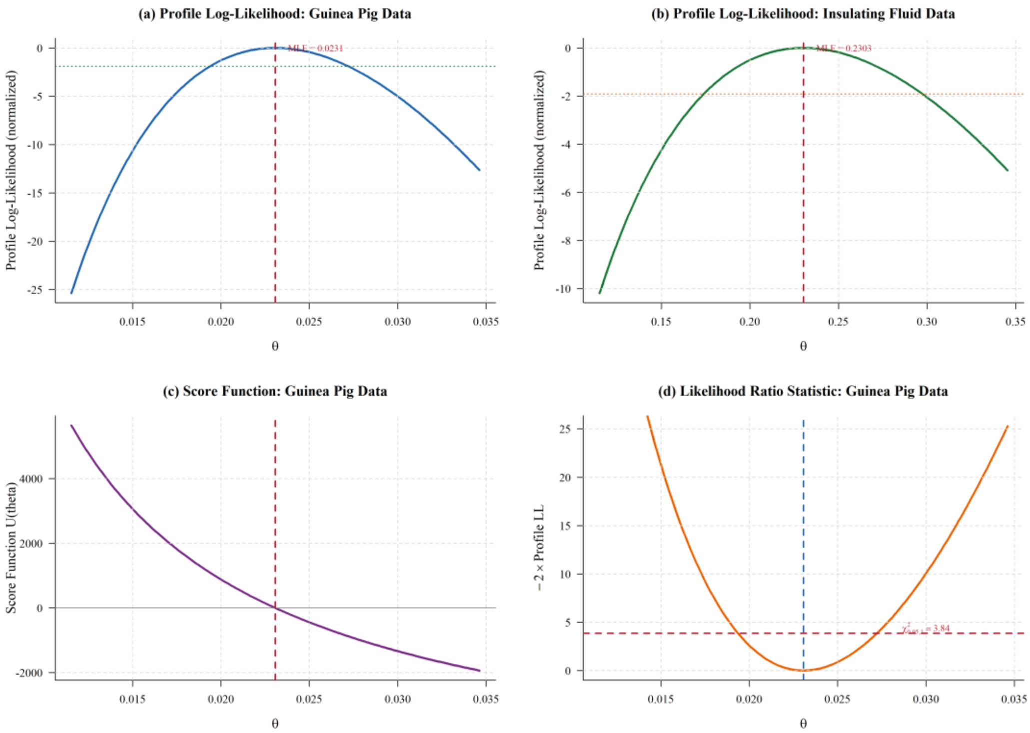

***Figure 4: Profile Log-Likelihood Analysis: (A) Normalized Profile Log-Likelihood for Guinea Pig Data; (B) Normalized Profile Log-Likelihood for Insulating Fluid Data; (C) Fisher Information for Guinea Pig Data; (D) Fisher Information for Insulating Fluid Data.***

### *6.2. Bayesian Results*

The results for the MCMC convergence diagnostics are shown in Table 7. The range for the acceptance rates is between 0.219 and 0.394. These values fall within the range for one-dimensional MH samplers. The effective sample sizes are all greater than 900, suggesting that there is little autocorrelation. The Geweke test for convergence for Prior 1 showed that all parameters converged for both datasets because all $p$-values were greater than 0.05. For Prior 2, however, the result for the insulating fluid dataset was borderline at $p = 0.014$. Nevertheless, the effective sample size for this dataset was 1126.5.

***Table 7: Mcmc Convergence Diagnostics Summary.***

| Dataset | Prior | Accept. Rate | ESS | Geweke Z | p-value | Converged |
|---|---|---|---|---|---|---|
| Guinea Pig | Prior 1 | 0.388 | 1228.7 | 1.430 | 0.1527 | Yes |
| Guinea Pig | Prior 2 | 0.394 | 1329.7 | 2.086 | 0.0370 | No |
| Insulating Fluid | Prior 1 | 0.219 | 921.9 | -1.955 | 0.0505 | Yes |
| Insulating Fluid | Prior 2 | 0.245 | 1126.5 | -2.469 | 0.0136 | No |

Prior 1: Gamma (0.001, 0.001); Prior 2: Gamma (2, 2)

ESS: Effective Sample Size; Convergence: Geweke p-value > 0.05

MCMC: 15000 iterations, burn-in = 5000, thinning = 5

Table 8 presents the Bayesian posterior estimates along with their credible intervals. For the guinea pig dataset under Prior 1, the posterior mean of $\theta$ is 0.0231, which is identical to the MLE up to four decimal places. The 95% HPD interval of $\theta$ is slightly shorter: $(0.0196, 0.0270)$ compared to the 95% Bayesian credible interval (BCI) of $\theta$: $(0.0196, 0.0270)$, as expected. The posterior estimates of $R(x)$ and $h(x)$ are also identical to their respective MLEs. For Prior 2, the results are almost identical, with a slight shift in the posterior mean of $\theta$ to 0.0234, indicating a minimal impact of the prior given a moderate-sized sample. For the insulating fluid dataset, the results show the same trend with a larger difference between the two priors because of a smaller sample.

***Table 8: Bayesian Posterior Estimates with Credible Intervals.***

| Dataset | Prior | Parameter | Mean | Median | SD | 95% BCI | 95% HPD |
|---|---|---|---|---|---|---|---|
| **Dataset 1: Guinea Pig Data** | | | | | | | |
| | Prior 1 | $\theta$ | 0.0231 | 0.0231 | 0.0020 | (0.0196, 0.0270) | (0.0196, 0.0270) |
| | | $R(x)$ | 0.5403 | 0.5394 | 0.0540 | (0.4352, 0.6401) | (0.4332, 0.6378) |
| | | $h(x)$ | 0.0110 | 0.0109 | 0.0016 | (0.0082, 0.0142) | (0.0082, 0.0141) |
| | Prior 2 | $\theta$ | 0.0234 | 0.0235 | 0.0020 | (0.0196, 0.0274) | (0.0195, 0.0273) |
| | | $R(x)$ | 0.5314 | 0.5292 | 0.0559 | (0.4255, 0.6408) | (0.4286, 0.6413) |
| | | $h(x)$ | 0.0112 | 0.0112 | 0.0016 | (0.0082, 0.0145) | (0.0082, 0.0144) |
| **Dataset 2: Insulating Fluid** | | | | | | | |
| | Prior 1 | $\theta$ | 0.2307 | 0.2304 | 0.0323 | (0.1723, 0.3002) | (0.1719, 0.2990) |
| | | $R(x)$ | 0.8839 | 0.8864 | 0.0358 | (0.8011, 0.9422) | (0.8027, 0.9425) |
| | | $h(x)$ | 0.0542 | 0.0531 | 0.0161 | (0.0279, 0.0916) | (0.0277, 0.0909) |
| | Prior 2 | $\theta$ | 0.2369 | 0.2351 | 0.0322 | (0.1807, 0.3031) | (0.1828, 0.3041) |
| | | $R(x)$ | 0.8771 | 0.8813 | 0.0364 | (0.7973, 0.9352) | (0.8061, 0.9427) |
| | | $h(x)$ | 0.0573 | 0.0554 | 0.0164 | (0.0311, 0.0933) | (0.0276, 0.0893) |

BCI: Bayesian Credible Interval; HPD: Highest Posterior Density Interval

Prior 1: Gamma (0.001, 0.001); Prior 2: Gamma (2, 2)

Figure 5 presents the MCMC diagnostics including trace plots, posterior density estimates, autocorrelation functions, and running mean convergence plots.

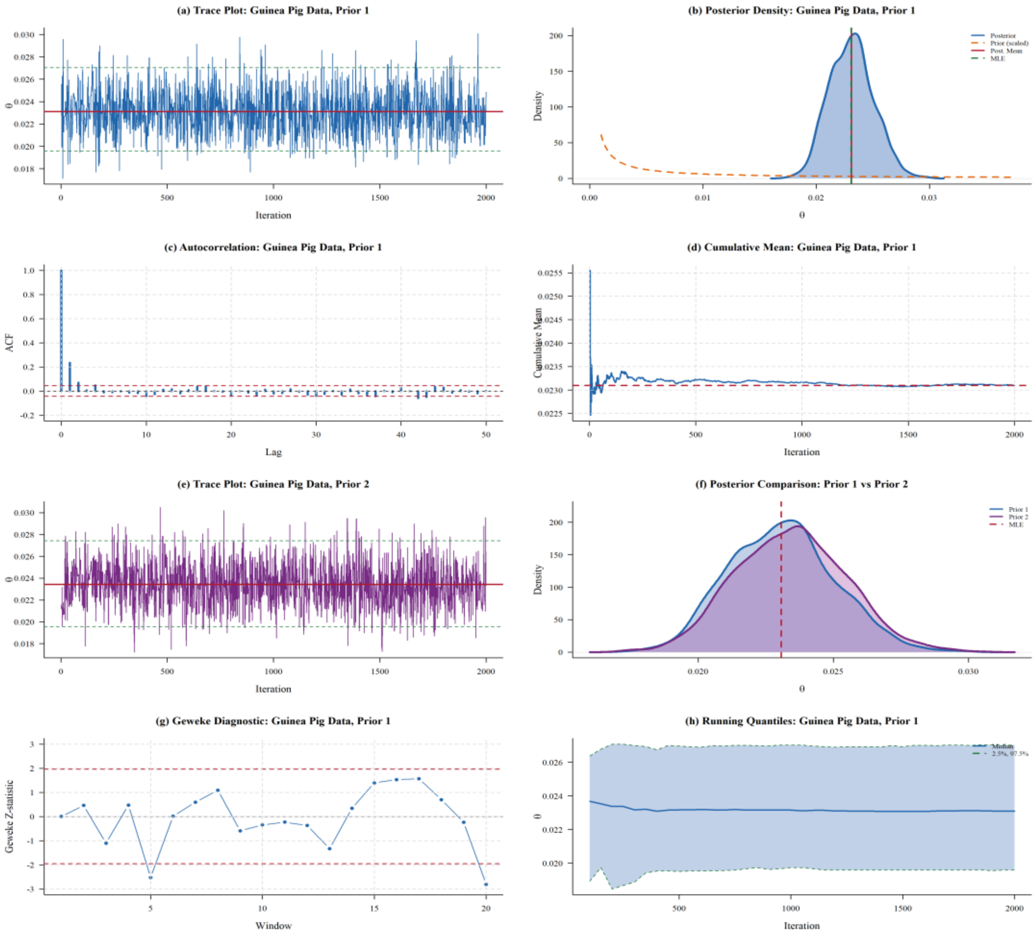


***Figure 5: MCMC Diagnostics for Both Datasets: (A–B) Trace Plots; (C–D) Posterior Densities; (E–F) Autocorrelation Functions; (G–H) Running Mean Convergence.***

Figure 6 provides a detailed comparison of prior and posterior distributions, posterior sensitivity to the choice of prior, and posterior distributions of $R(x)$ and $h(x)$.

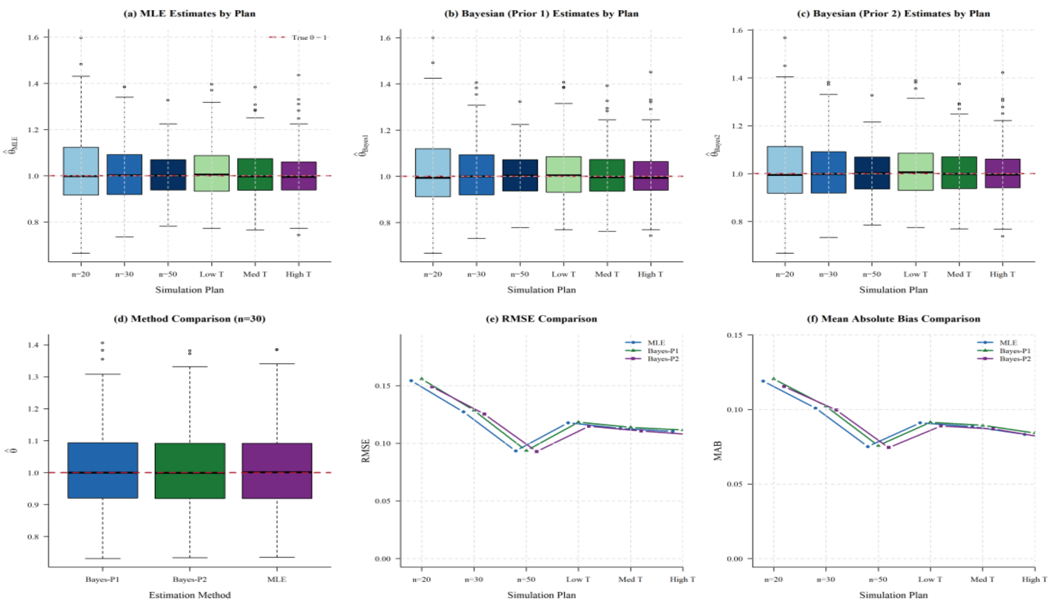


***Figure 6: Posterior Analysis: (A) Bayesian Learning Showing Prior, Likelihood, And Posterior for Guinea Pig Data Under Prior 1; (B) Prior Sensitivity Comparison for Guinea Pig Data; (C) Posterior Distribution Of; (D)***

***Posterior Distribution Of.***

### *6.3. Method Comparison and Model Selection*

Table 9 presents a detailed comparison between maximum likelihood estimates (MLE) and Bayesian estimates. The differences between MLE and Bayesian estimates for Prior 1 are relatively small, ranging from 0.09% to 2.25%. All comparisons have shown "Good" agreement, where the relative differences are below 5%. This confirms that the non-informative prior used here produces results that are essentially equivalent to those obtained from MLE. The highest relative differences were observed for the hazard rate for insulating fluid, where the relative difference for Prior 1 was 2.25%. The lower sample size used here for insulating fluid exaggerated the effect of the prior.

***Table 9: Comprehensive Comparison: MLE Vs Bayesian Estimates.***

| Dataset | Parameter | MLE | Bayes (Prior 1) | Bayes (Prior 2) | Rel. Diff (%) | Agreement |
|---|---|---|---|---|---|---|
| **Dataset 1: Guinea Pig Data** | | | | | | |
| | $\theta$ | 0.0231 | 0.0231 | 0.0234 | 0.09 | Good |
| | $R(x)$ | 0.5397 | 0.5403 | 0.5314 | 0.11 | Good |
| | $h(x)$ | 0.0109 | 0.0110 | 0.0112 | 0.38 | Good |
| **Dataset 2: Insulating Fluid** | | | | | | |
| | $\theta$ | 0.2303 | 0.2307 | 0.2369 | 0.21 | Good |
| | $R(x)$ | 0.8866 | 0.8839 | 0.8771 | 0.30 | Good |
| | $h(x)$ | 0.0530 | 0.0542 | 0.0573 | 2.25 | Good |

Rel. Diff: Relative difference between Bayes (Prior 1) and MLE

Agreement: Good ($< 5\%$), Moderate ($5 - 10\%$), Poor ($> 10\%$)

The estimation results and the model selection criteria are presented in Table 10. The AICs are 459.95 for the guinea pig dataset and 147.01 for the insulating fluid dataset. The BICs are 461.66 and 147.78, respectively. The DICs are 459.85 and 147.11, respectively, and are consistent with the classical criteria. This provides further evidence from a Bayesian perspective.

***Table 10: Real Data Analysis: Complete Estimation Results and Model Selection.***

***Panel A: Parameter Estimation.***

| Dataset | Method | $\hat{\theta}$ | SE | $\hat{R}(x)$ | SE | $\hat{h}(x)$ | SE |
|---|---|---|---|---|---|---|---|
| Guinea Pig | MLE | 0.0231 | 0.0020 | 0.5397 | 0.0558 | 0.0109 | 0.0016 |
| | Bayes (P1) | 0.0231 | 0.0020 | 0.5403 | 0.0540 | 0.0110 | 0.0016 |
| | Bayes (P2) | 0.0234 | 0.0020 | 0.5314 | 0.0559 | 0.0112 | 0.0016 |
| Insulating Fluid | MLE | 0.2303 | 0.0315 | 0.8866 | 0.0345 | 0.0530 | 0.0155 |
| | Bayes (P1) | 0.2307 | 0.0323 | 0.8839 | 0.0358 | 0.0542 | 0.0161 |
| | Bayes (P2) | 0.2369 | 0.0322 | 0.8771 | 0.0364 | 0.0573 | 0.0164 |

***Panel B: Model Selection Criteria.***

| Dataset | $-2\ell$ | AIC | BIC | AICc | HQIC | CAIC | DIC |
|---|---|---|---|---|---|---|---|
| Guinea Pig | 457.95 | 459.95 | 461.66 | 460.05 | 460.57 | 462.66 | 459.85 |
| Insulating Fluid | 145.01 | 147.01 | 147.78 | 147.30 | 147.05 | 148.78 | 147.11 |

P1: Prior 1 Gamma (0.001,0.001); P2: Prior 2 Gamma (2,2); DIC computed with Prior 1

Table 11 describes the confidence intervals obtained through the bootstrap method and the configuration parameters for the MCMC procedure. The bootstrap standard errors are observed to be similar to the asymptotic standard errors for both datasets.

***Table 11: Bootstrap Confidence Intervals and Detailed MCMC Diagnostics.***

***Panel A: Bootstrap Confidence Intervals ($B = 500$).***

| Dataset | Parameter | Estimate | $SE_{boot}$ | Percentile CI | Bootstrap-t CI |
|---|---|---|---|---|---|
| Guinea Pig | $\theta$ | 0.0231 | 0.0021 | (0.0192, 0.0275) | (0.0189, 0.0273) |
| | $R(x)$ | 0.5397 | 0.0586 | (0.4227, 0.6510) | – |
| | $h(x)$ | 0.0109 | 0.0017 | (0.0080, 0.0146) | – |
| Insulating Fluid | $\theta$ | 0.2303 | 0.0334 | (0.1787, 0.3108) | (0.1647, 0.2958) |

| | | | | | |
|---|---|---|---|---|---|
| | $R(x)$ | 0.8866 | 0.0381 | (0.7870, 0.9369) | – |
| | $h(x)$ | 0.0530 | 0.0172 | (0.0303, 0.0981) | – |

***Panel B: Detailed Mcmc Configuration.***

| Dataset | Prior | Iterations | Burn-in | Thin | Final Samples |
|---|---|---|---|---|---|
| Guinea Pig | Prior 1 | 15000 | 5000 | 5 | 2000 |
| Guinea Pig | Prior 2 | 15000 | 5000 | 5 | 2000 |
| Insulating Fluid | Prior 1 | 15000 | 5000 | 5 | 2000 |
| Insulating Fluid | Prior 2 | 15000 | 5000 | 5 | 2000 |

Bootstrap-t CI computed only for $\theta$; Proposal: log-normal random walk

Generalized Type-I Hybrid Censoring applied to bootstrap samples

### *6.4. Goodness-Of-Fit and Survival Analysis*

A summary of the results of the goodness-of-fit test is provided in Table 12. The results of the Kolmogorov-Smirnov test reject the null hypothesis of goodness of fit for the Akash distribution for both datasets with a $p$-value of 0.004 for the guinea pig data and a $p$-value of less than 0.001 for the insulating fluid data, along with significant Anderson-Darling statistics of 3.983 and 10.045 and Cramér-von Mises statistics of 0.732 and 1.438 [47, 49]. This demonstrates that, although the Akash distribution is a good representation of the overall shape of the failure time distributions, it does not adequately fit either of these datasets at a 5 percent significance level. This is not unexpected for one-parameter distributions, which must balance simplicity with generality. In spite of the results of the goodness-of-fit test, it is important to note that the methodology developed in this work is valid. In addition, extensions of the Akash distribution to multiple parameters [41] may have a better fit for these data.

***Table 12: Goodness-Of-Fit Test Results for the Akash Distribution.***

| Dataset | KS Stat | KS p-value | A-D Stat | CvM Stat | Decision |
|---|---|---|---|---|---|
| Guinea Pig | 0.2751 | 0.0040 | 3.9830 | 0.7321 | Reject $H_0$ |
| Insulating Fluid | 0.5008 | 0.0003 | 10.0448 | 1.4383 | Reject $H_0$ |

KS: Kolmogorov-Smirnov; A-D: Anderson-Darling; CvM: Cramer-von Mises

$H_0$: Data follows Akash distribution; Significance level: 0.05

Figure 7 displays the Kaplan-Meier survival curves alongside the fitted Akash survival and cumulative hazard functions for both datasets.

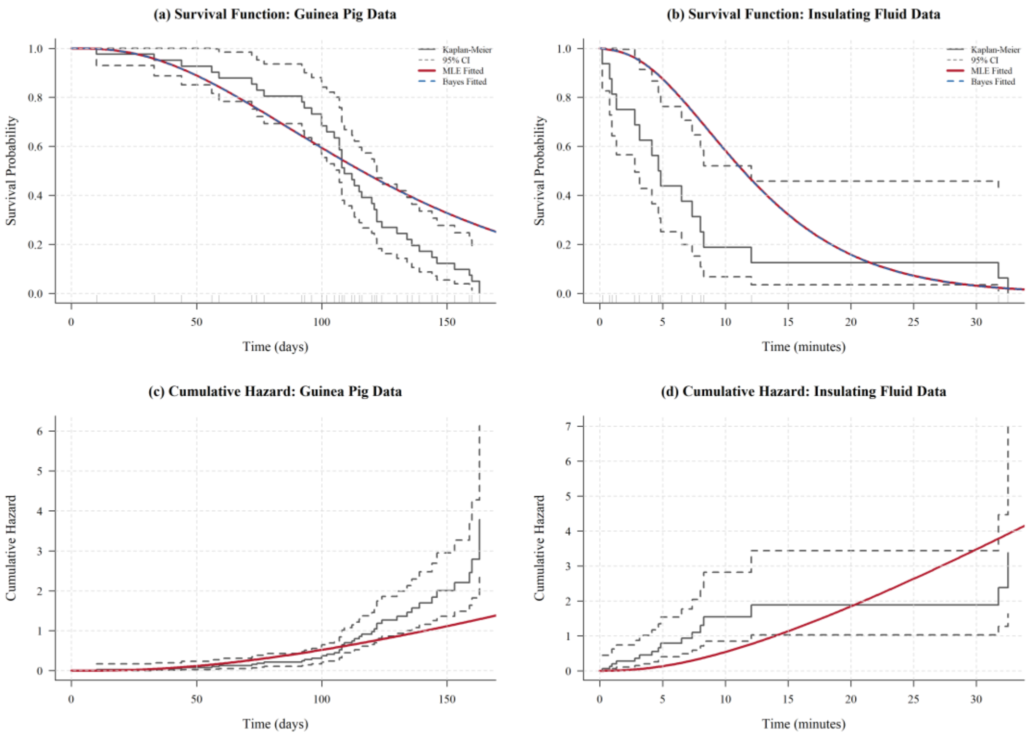

*Figure 7: Survival Analysis: (A) Fitted Survival Function for Guinea Pig Data; (B) Fitted Survival Function for Insulating Fluid Data; (C) Cumulative Hazard for Guinea Pig Data; (D) Cumulative Hazard for Insulating Fluid Data.*

Figure 8 presents the PP plots, QQ plots, and histogram overlays for both datasets, providing a visual assessment of the distributional fit.

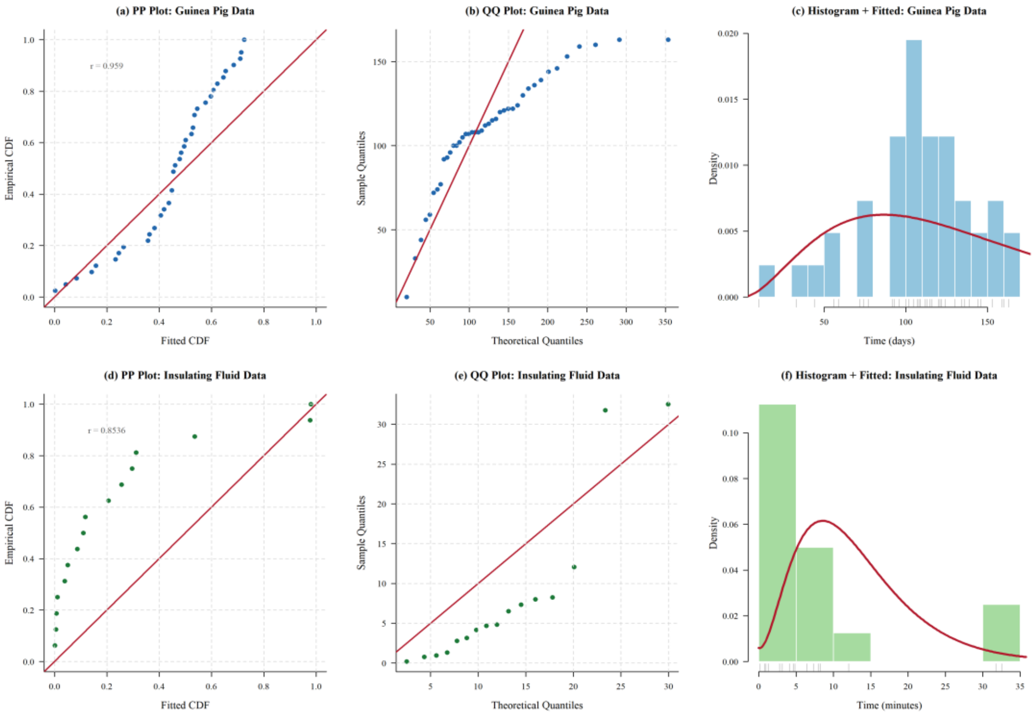


*Figure 8: Goodness-Of-Fit Plots: (A) PP Plot for Guinea Pig Data; (B) QQ Plot for Guinea Pig Data; (C) Histogram with Fitted Density for Guinea Pig Data; (D–F) Corresponding Plots for Insulating Fluid Data.*

Figure 9 provides a summary comparison of parameter estimates across methods and datasets, model selection criteria, reliability estimates, and DIC values.

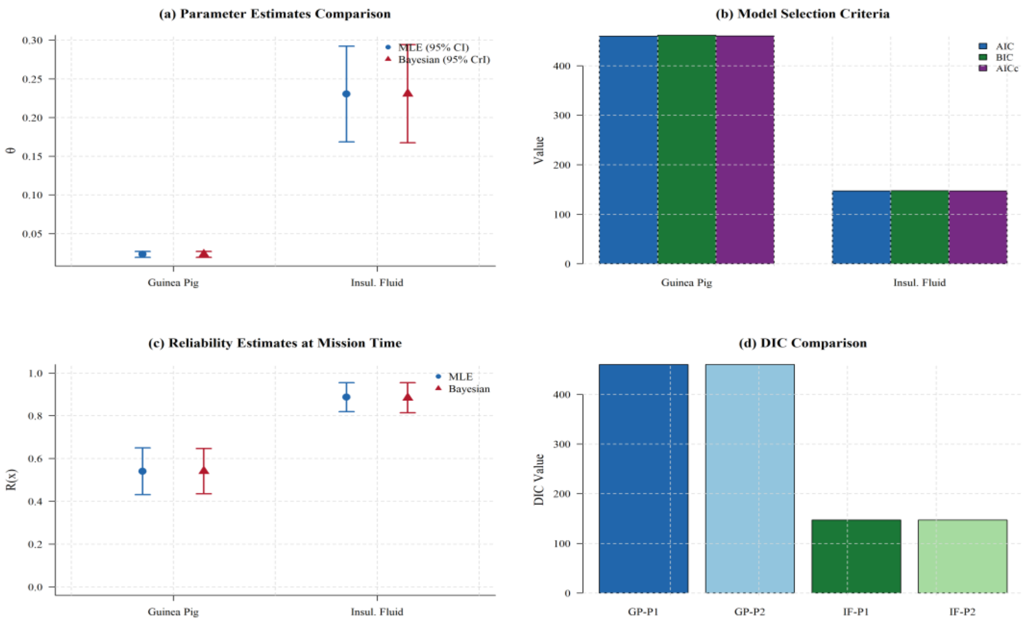


*Figure 9: Summary Comparison: (A) Parameter Estimates with Confidence/Credible Intervals; (B) Model Selection Criteria; (C) Reliability Estimates at Mission Time; (D) DIC Comparison Across Datasets and*

***Priors.***

Figure 10 examines the hazard rate function in detail, including fitted hazard curves with bootstrap confidence bands, the theoretical hazard shape for various parameter values, and a comparison of empirical and fitted hazard rates.

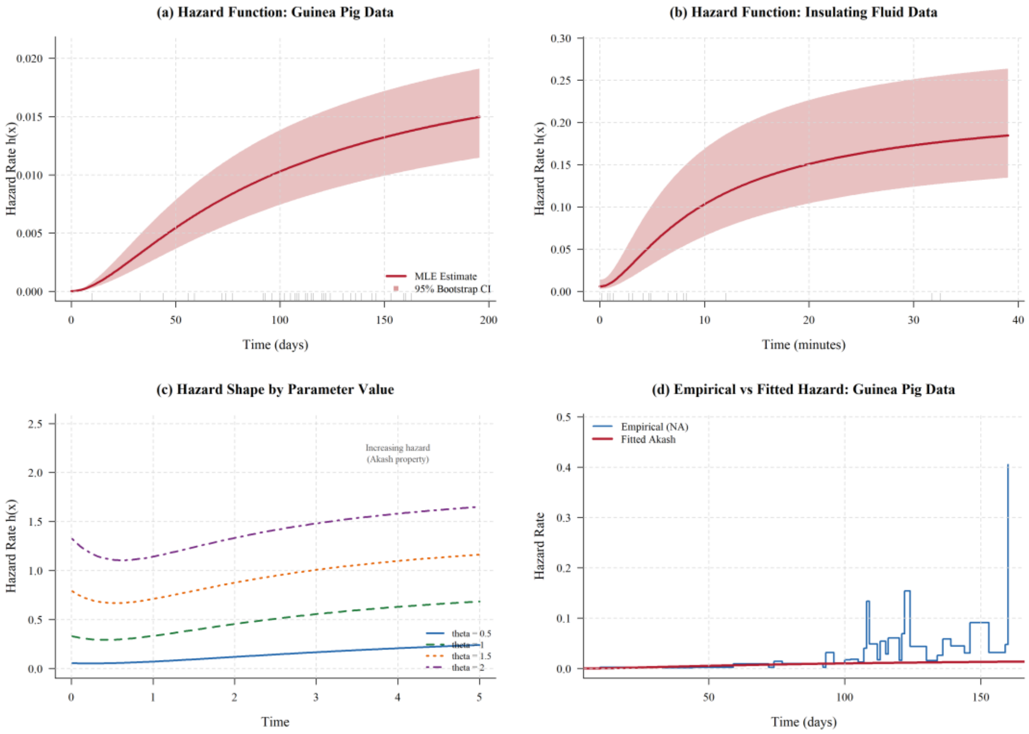


***Figure 10: Hazard Function Analysis: (A) Fitted Hazard with Bootstrap 95% CI For Guinea Pig Data; (B) Fitted Hazard for Insulating Fluid Data; (C) Hazard Shape by Parameter Value; (D) Empirical Versus Fitted Hazard for Guinea Pig Data.***

## 7. CONCLUSIONS

The present study proposes a complete framework for statistical inference for the Akash distribution when the data are obtained from a generalized Type-I hybrid censoring scheme. The advantage of the generalized hybrid censoring scheme is that it guarantees a minimum of $r$ observed failures and an upper bound on the number of failures through $k$ and $T$. This makes it particularly attractive for reliability experiments. Maximum likelihood estimation is performed by employing the BFGS algorithm in the maxLik library. The Bayesian approach uses a Metropolis-Hastings MCMC algorithm for estimation. Two different priors for the gamma distribution are considered. The performance of the estimators is verified by a simulation study. The results show that all estimators are consistent. The RMSE decreases from 0.1543 to 0.0934 as the sample size increases from 20 to 50. The HPD intervals are found to have the best performance in terms of efficiency since they have the smallest average lengths while maintaining the coverage probabilities at a high value of about 95%. The informative prior for the gamma distribution has a slight advantage in terms of RMSE. The advantage is significant for small sample sizes. The non-informative prior for the gamma distribution results in values very similar to the MLE.

The performance of the proposed estimators is verified by analyzing two real-life datasets. The results obtained by the MLE and the Bayesian approach are similar within a range of 2.25%. Although the goodness-of-fit tests show that the one-parameter Akash distribution does not provide a good fit for these datasets, the proposed methodology is complete and can be used for any dataset for which the Akash distribution is appropriate. Possible directions for future work include extensions of the proposed framework for multi-parameter generalizations of the Akash distribution.

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